UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2015 1:30 pm to 3:30 pm

PAPER 11

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the LYM inequality.

Show that the n-cube $\mathbb{P}[n]$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ symmetric chains.

Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}[n]$ be two antichains. Suppose that $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Show that $|\mathcal{A}| + |\mathcal{B}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$, or give a counterexample.

2

State and prove the Kruskal-Katona theorem.

Let $\mathcal{I}$ be an initial segment of colex order on $\mathbb{N}^{(r+1)}$. Show that, if $|\mathcal{I}| \geqslant \binom{m}{r+1} - m + r + 1$, then $[m]^{(r)} \subset \partial \mathcal{I}$.

Let $\mathcal{A} \subset \mathbb{P}[n]$ be such that $|A| \geqslant r$ for all $A \in \mathcal{A}$. Let $\mathcal{B} = \{Y \in [n]^{(r)} : Y \subset A \text{ for some } A \in \mathcal{A}\}$. Show that, if

$$|\mathcal{A}| > \sum_{j=r}^{m} \binom{m}{j} - \binom{m-r}{j-r}$$

for some integer m, then $|\mathcal{B}| \geqslant \binom{m}{r}$.

3

Let $\mathcal{A} \subset \mathcal{P}[n]$ be a 2-intersecting system. Show that

$$|\mathcal{A}| \leqslant \sum_{i=n/2+1}^{n} \binom{n}{i} \qquad \text{if } n \text{ is even}$$

$$|\mathcal{A}| \leqslant \binom{n-1}{(n+1)/2} + \sum_{i=(n+3)/2}^{n} \binom{n}{i} \qquad \text{if } n \text{ is odd},$$

and show that equality can be attained. (You may assume the Erdős-Ko-Rado theorem.)

Show that if $\mathcal{A} \subset [n]^{(r)}$ is r-uniform and 2-intersecting, then $|\mathcal{A}| \leqslant \binom{r}{2}\binom{n-2}{r-2}$.

Show that if $\mathcal{A} \subset [n]^{(r)}$ is r-uniform and intersecting, then there exists $x \in [n]$ such that at most $(r-1)^2\binom{n-2}{r-2}$ sets in $\mathcal{A}$ do not contain x.

4

State and prove the Frankl-Wilson theorem, concerning set families whose intersection sizes are constrained modulo some prime.

Deduce, or prove directly, that if $\mathcal{A} \subset \mathcal{P}[n]$ and $|A \cap B| = \lambda$ for all distinct $A, B \in \mathcal{A}$, then $|\mathcal{A}| \leqslant n$, unless $\lambda = 0$ in which case $|\mathcal{A}| \leqslant n+1$.

Let $t \geqslant 3$ and let $A_1, \ldots, A_m$ be distinct subsets of $[n]$ such that, for all sets $\{i_1, \ldots, i_t\} \in [m]^{(t)}$ of t distinct indices, $|A_{i_1} \cap \cdots \cap A_{i_t}| = \lambda$ holds. Show that either

(a) there exists some $S \subset [n]$ with $|S| = \lambda$ and $S \subset A_i$ for every $i \in [m]$, or

(b) $m \leqslant k + t - 2$, where $k = \min\{|A_{i_1} \cap \cdots \cap A_{i_{t-2}}| : \{i_1, \ldots, i_{t-2}\} \in [m]^{(t-2)}\}$.

Show that the bound in case (b) cannot be improved in general.

END OF PAPER