UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 1:30 pm to 4:30 pm

PAPER 10

INTRODUCTION TO NONLINEAR WAVE EQUATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In the first two parts of this problem, consider the linear homogeneous wave equation

$$\Box\phi = 0$$

in $\mathbb{R} \times \mathbb{R}^3$. (Here, $\Box := -\partial_t^2 + \sum_{i=1}^3 \partial_{x_i}^2$.)

1. State the finite speed of propagation and the strong Huygen's principle.
2. Give a proof of the finite speed of propagation.

We now turn to the linear inhomogeneous wave equation

$$\Box\phi = F$$

in $\mathbb{R} \times \mathbb{R}^3$ in spherical symmetry.

3. Let $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $v = t + r$ and $u = t - r$. Assuming that ϕ and F are both spherically symmetric, write down the equation in the v, u coordinates.
4. Assume that F is smooth, spherically symmetric, supported in the set $\cup_t(\{t\} \times (B(0, t+1) \setminus B(0,t)))$ and obey the estimate

   $$\sup_x |F|(t,x) \leqslant \frac{1}{(1+t)^2}.$$

   Suppose the initial data for ϕ are smooth and compactly supported in $B(0,1)$. Show that there exists $C > 0$ (depending on the initial data) such that

   $$\sup_{\{x: t+\frac{1}{2} \leqslant |x| \leqslant t+1\}} |\phi|(t,x) \leqslant \frac{C\log(2+t)}{1+t}.$$

   (You can use any statements from any of the previous parts of this problem.)

2

1. State a version of the Klainerman-Sobolev inequality in $\mathbb{R} \times \mathbb{R}^3$. State clearly the set of vector fields that are used in this inequality.

2. Consider the nonlinear wave equation

$$\Box\phi = (\partial_t \phi)^2$$

with initial data

$$(\phi, \partial_t \phi) \restriction_{\{t=0\}} = (\epsilon\phi_0, \epsilon\phi_1) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

(Here, $\Box := -\partial_t^2 + \sum_{i=1}^3 \partial_{x_i}^2$.) Show that for every integer $N \geqslant 1$, there exists $\epsilon_N > 0$ depending only on ϕ_0, ϕ_1 and N such that if $\epsilon \leqslant \epsilon_N$, then the solution exists and remains smooth in the time interval $[0, \epsilon^{-N}]$. (You can use the standard energy estimates for the linear inhomogeneous wave equation without proof and assume results on local existence of solutions.)

3

Consider the nonlinear wave equation for $\phi : I \times \mathbb{R}^3 \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval)

$$\Box\phi = \phi^3$$

with smooth and compactly supported initial data, i.e.,

$$(\phi, \partial_t\phi) \restriction_{\{t=0\}} = (\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

(Here, $\Box := -\partial_t^2 + \sum_{i=1}^3 \partial_{x_i}^2$.)

1. Show that an appropriately defined positive energy is conserved.

2. It is known that for the given initial data, there exists a unique smooth solution which is global in time. One of the estimates that can be used to obtain this result is the following: Given initial data such that

$$\|\phi_0\|_{H^2(\mathbb{R}^3)} + \|\phi_1\|_{H^1(\mathbb{R}^3)} \leqslant D$$

and suppose the initial value for the energy defined in the previous part of this problem is $E \in [0, \infty)$. Then there exists a continuous and locally bounded function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ depending only on D and E such that the following bound is obeyed as long as the solution exists:

$$\sup_{t\in[0,T]} (\|\phi\|_{\dot{H}^2(\mathbb{R}^3)}(t) + \|\partial_t\phi\|_{\dot{H}^1(\mathbb{R}^3)}(t)) \leqslant f(T).$$

Prove this estimate assuming that a smooth solution to the equation exists. (You can use the energy estimates for the inhomogeneous wave equation without proof. You can also use the Sobolev embedding theorem without proof, as long as you correctly state the version of the theorem that you are using.)

3. Consider now instead the nonlinear equation with an opposite sign:

$$\Box\phi = -\phi^3.$$

Show that there exist solutions arising from smooth and compactly supported initial data which blow up in finite time. (Hint: First, look for solutions of the form $\frac{\beta}{(t-1)^\alpha}$. Then modify it appropriately so that the initial data are smooth and compactly supported.)

4

Write an essay on the wave map equations for maps $\phi : I \times \mathbb{R}^n \to \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x|^2 = 1\} \subset \mathbb{R}^3$. The system of equations is given by

$$\Box\phi = \phi(\partial_t\phi^t\partial_t\phi - \sum_{i=1}^{n} \partial_{x_i}\phi^t\partial_{x_i}\phi).$$

(Here, $\Box := -\partial_t^2 + \sum_{i=1}^n \partial_{x_i}^2$.) Discuss briefly (in no more than 80 words each) **at least 5** of the following aspects (It is possible to get full marks by discussing exactly 5 of the following aspects, but you can also potentially obtain higher marks if you explain more than 5 of these. Main results and ideas should be stated, but details of the proofs are not expected.):

1. What are the appropriate initial condition.
2. Local existence.
3. Existence of non-constant global solutions which are independent of time in the case $n = 2$.
4. Scaling, conservation law and criticality.
5. Global regularity in $(1 + 1)$-dimensions.
6. Global regularity for small and localized initial data in $(4 + 1)$-dimensions.
7. Global regularity for small and localized initial data in $(3 + 1)$-dimensions.

END OF PAPER