UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2015 9:00 am to 12:00 pm

PAPER 1

COMMUTATIVE ALGEBRA

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

Except in question 6, all rings are assumed to be commutative.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define what it means for a ring R to be Noetherian.

Let R be a Noetherian ring and let f be a ring homomorphism from R to R. Show that if f is surjective than it is also injective.

Define the nilradical $N(R)$ of a ring R. Show that it is equal to the intersection of all the prime ideals of R, and, if R is Noetherian, that it is the intersection of finitely many prime ideals.

Show, for any ring R, that $N(R) = 0$ if and only if $N(R[[X]]) = 0$.

Give an example of a ring whose nilradical is not nilpotent.

2

(i) What does it mean for an ideal of a ring R to be prime?

Let P_1, P_2 and P_3 be prime ideals of R with $P_i \leqslant P_j$ only if $i = j$. Show that their union U is not an ideal.

Give an example to show that if the ideals P_i are not necessarily prime then their union may be an ideal.

(ii) What does it mean for a subset S of R to be multiplicatively closed?

Let S be a multiplicatively closed subset of R. Show that the set of prime ideals of the localisation $S^{-1}R$ corresponds bijectively to the set of prime ideals of R disjoint to S.

(iii) A multiplicatively closed subset S is said to be saturated if $xy \in S$ only if both $x \in S$ and $y \in S$. Show that a multiplicatively closed subset of R is saturated if and only if its complement in R is a union of prime ideals.

Give an example of a ring R and a saturated multiplicatively closed subset S of R whose complement in R is not a union of finitely many prime ideals.

3

(i) What does it mean for a ring T to be integral over a subring R?

Suppose T is integral over R. Let P be a prime ideal of R. Show that there is a prime ideal Q of T with $Q \cap R = P$.

(ii) Let k be a field and let I be an ideal of the Laurent polynomial algebra $A = k[X_1, X_1^{-1}, X_2, X_2^{-1}, X_3, X_3^{-1}]$. Let $T = A/I$. Show that if T is not finite dimensional as a k-vector space then it is integral over some subring isomorphic to a Laurent polynomial algebra $k[Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$ for some m with $1 \leqslant m \leqslant 3$.

4

Define what is meant by the height of a prime ideal P of a Noetherian ring R.

Let I be a proper ideal generated by n elements. Show that for each minimal prime P over I the height of P is at most n.

5

What does it mean for a ring R to be local?

Define dim(R) and $d(R)$ for a Noetherian local ring R and show that $\dim(\mathrm{R}) \leqslant \mathrm{d(R)}$.

What does it mean for a Noetherian local ring to be regular?

Show that if R is a regular Noetherian local ring then it is an integral domain.

6

Let k be a field. Define the ring of differential operators of a commutative k-algebra R. Define the set Der(R) of derivations of R.

What is meant by the order of a differential operator?

Show that the set of differential operators of order at most 1 is equal to $R + \mathrm{Der(R)}$.

Define the nth complex Weyl algebra A_n and show that every non-zero A_n-module has infinite dimension as a complex vector space.

Define $d(M)$ for a non-zero finitely generated A_n-module M, and show that $d(M)$ is at least n.

END OF PAPER