UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 29 May, 2014 9:00 am to 11:00 am

PAPER 9

RAMSEY THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Show that, whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < \ldots$ such that the set $\{x_i + 2x_j : \ i < j\}$ is monochromatic.

(ii) Show that it is *not* true that, whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < \ldots$ such that the set

$$\{x_i + 2x_j : \ i < j\} \cup \{x_i + x_j : \ i < j\}$$

is monochromatic.

(iii) Show that, whenever $\mathbb{N}$ is finitely coloured, there exist $x_1 < x_2 < \ldots$ and $y_1 < y_2 < \ldots$ such that the set

$$\{x_i + 2x_j : \ i < j\} \cup \{y_i + y_j : \ i < j\}$$

is monochromatic.

[*Hint: Find a sequence* $u_1 < u_2 < \ldots$ *such that the set*

$$\{u_i + u_j + 2u_k + 2u_l : \ i < j < k < l\}$$

is monochromatic, and then construct the x_i *and the* y_i *from this sequence.*]

2

State and prove Rado's theorem.

[You may assume that, for any m, p, c, whenever $\mathbb{N}$ is finitely coloured there is a monochromatic (m, p, c)-set.]

By considering the system of equations $3x + y = 3z, x - 2y = rw$ (in variables x, y, z, w), for a suitable rational number r, show that, whenever $\mathbb{N}$ is finitely coloured, there exist monochromatic x, y, z with $3x + y = 3z$ and $2y < x$.

3

What is an *ultrafilter* on $\mathbb{N}$? Show that there exists a non-principal ultrafilter on $\mathbb{N}$. Define the topological space $\beta\mathbb{N}$, and prove that it is compact and Hausdorff.

State Hindman's theorem, and show how to deduce it from the existence of an idempotent for $+$ on $\beta\mathbb{N}$. (You are not required to prove that an idempotent exists. You may assume simple properties of ultrafilters, their quantifiers, and the operation $+$ on $\beta\mathbb{N}$.)

For a filter $\mathcal{F}$ on $\mathbb{N}$, and a statement p, we write $\forall_{\mathcal{F}} x\ p(x)$ to mean that

$$\{x \in \mathbb{N} :\ p(x)\} \in \mathcal{F}\ .$$

Which of the following are always true and which can be false (for statements p and q and filters $\mathcal{F}$ and $\mathcal{G}$)?

(i) $\forall_{\mathcal{F}} x\ (p(x)$ and $q(x))$ if and only if $(\forall_{\mathcal{F}} x\ p(x)$ and $\forall_{\mathcal{F}} x\ q(x))$

(ii) $\forall_{\mathcal{F}} x\ (p(x)$ or $q(x))$ if and only if $(\forall_{\mathcal{F}} x\ p(x)$ or $\forall_{\mathcal{F}} x\ q(x))$

(iii) $(\forall_{\mathcal{F}} x\ p(x))$ is false if and only if $\forall_{\mathcal{F}} x$ (not $p(x))$

(iv) If we define $\mathcal{F}+\mathcal{G}$ to be the collection of all $A \subset \mathbb{N}$ such that $\forall_{\mathcal{F}} x\ \forall_{\mathcal{G}} y\ (x+y \in A)$, then $\mathcal{F}+\mathcal{G}$ is a filter.

4

What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *Ramsey*? Give an example of a non-Ramsey set. Prove that every $*$-open set is Ramsey.

[Any results quoted from the course must be proved.]

Give an example of a τ-Baire set that is not Ramsey.

END OF PAPER