UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 10 June, 2014 9:00 am to 12:00 pm

PAPER 81

QUANTUM CONDENSED MATTER FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The one-dimensional spin S quantum Heisenberg Ferromagnet is specified by the Hamiltonian,

$$\hat{H} = -J \sum_{n=1}^{N} \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1} ,$$

where $J > 0$, and periodic boundary conditions are imposed such that $\hat{\mathbf{S}}_{N+1} = \hat{\mathbf{S}}_1$. Specify one of the possible ferromagnetic ground states and explain how other ground states can be generated from it. What are the physical consequences of the ground state degeneracy for the low-energy spectrum of spin fluctuations?

(a) Show that the Holstein–Primakoff transformation,

$$\hat{S}^+ = (2S)^{1/2} \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2} a, \qquad \hat{S}^- = (\hat{S}^+)^\dagger, \qquad \hat{S}^z = S - a^\dagger a \, ,$$

is consistent with the quantum spin algebra of spin S.

(b) Taking the spin to be large $S \gg 1$, expand the ferromagnetic Hamiltonian, $\hat{H}$, around a ground state. Confirm that, to order S, the Hamiltonian can be written as a bilinear in the boson operators a and $a^\dagger$.

(c) Bringing the Hamiltonian to diagonal form, show that the excitation spectrum of the resulting Hamiltonian is given by

$$\omega_k = JS \sin^2(k/2) \, .$$

Define the quantum numbers k and comment on the form of the low-energy spectrum.

(d) Including an exchange interaction between next-nearest neighbour spins, the Hamiltonian takes the form,

$$\hat{H} = -\sum_{n=1}^{N} \left[J_1 \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1} + J_2 \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+2} \right] ,$$

where both $J_1 > 0$ and $J_2 > 0$. Once again, imposing periodic boundary conditions, $\hat{\mathbf{S}}_{N+1} = \hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_{N+2} = \hat{\mathbf{S}}_2$, obtain the spectrum of the Hamiltonian.

2

The weakly interacting Bose gas in a box of size L is defined by the Hamiltonian,

$$\hat{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}\, b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + \frac{1}{2L^3} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V_{\mathbf{q}}\, b^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k}'} b_{\mathbf{k}'+\mathbf{q}} b_{\mathbf{k}-\mathbf{q}}\,,$$

where the operators $b^\dagger_{\mathbf{k}}$ and $b_{\mathbf{k}}$ obey bosonic commutation relations, and $\epsilon_{\mathbf{k}} = \dfrac{\hbar^2 k^2}{2m}$ describes the free particle dispersion of particles with mass m.

(a) At low temperatures, a macroscopic fraction of particles condense into the non-interacting ($k = 0$) ground state. In this limit, show that the Hamiltonian takes the approximate form,

$$\hat{H} \simeq \frac{L^3}{2} V_0 n^2 + \sum_{\mathbf{k}\neq 0} \left[(\epsilon_{\mathbf{k}} + nV_{\mathbf{k}})\, b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + \frac{n}{2} \left(V_{\mathbf{k}} b^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} + \text{h.c.} \right) \right] ,$$

where $n = N/L^3$ denotes the particle density. Comment both on the physical origin of each of the terms in the expansion of the Hamiltonian, and the scale of the terms that have been neglected.

(b) By implementing an appropriate canonical transformation of the boson operators, show that the Hamiltonian can be brought to the diagonal form,

$$\hat{H} = \frac{L^3}{2} V_0 n^2 - \frac{1}{2} \sum_{\mathbf{k}\neq 0} (\epsilon_{\mathbf{k}} + nV_{\mathbf{k}}) + \sum_{\mathbf{k}\neq 0} E_{\mathbf{k}} \left(\alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + 1/2 \right) ,$$

where the operators $\alpha^\dagger_{\mathbf{k}}$ and $\alpha_{\mathbf{k}}$ obey bosonic commutation relations. Sketch and comment upon the low-energy form of the quasi-particle dispersion $E_{\mathbf{k}}$.

(c) Show that the density of particles outside the condensate is given by

$$n - n_0 = \frac{1}{2L^3} \sum_{\mathbf{k}\neq 0} \left(\frac{\epsilon_{\mathbf{k}} + nV_{\mathbf{k}}}{E_{\mathbf{k}}} - 1 \right) .$$

3

The quantum rotor is defined by the Hamiltonian,

$$\hat{H} = \frac{\hat{\ell}^2}{2I},$$

where $\hat{\ell} = -i\hbar\partial_\phi$ denotes the angular momentum operator and ϕ the angular coordinate.

(a) By finding the eigenstates of the rotor Hamiltonian, show that the quantum partition function $\mathcal{Z} = \mathrm{Tr}\, e^{-\beta\hat{H}}$ is given by the sum

$$\mathcal{Z} = \sum_{n=-\infty}^{\infty} \exp\left[-\beta\frac{\hbar^2 n^2}{2I}\right].$$

(b) Explain the physical meaning of the propagator, $\langle\phi_{\mathrm{F}}|\exp\left[-\frac{\mathrm{i}}{\hbar}\hat{H}t\right]|\phi_{\mathrm{I}}\rangle$. Starting from the Feynman path integral for the propagator, show that the partition function can also be expressed as the functional integral,

$$\mathcal{Z} = \int_0^{2\pi} d\phi \sum_{m=-\infty}^{\infty} \int_{\phi(0)=\phi,\, \phi(\beta)=\phi+2\pi m} D\phi(\tau) \exp\left[-\frac{1}{\hbar^2}\int_0^\beta d\tau\, \frac{I}{2}\dot{\phi}^2\right].$$

(c) By evaluating the path integral, show that

$$\mathcal{Z} = 2\pi\, \mathcal{Z}_0 \sum_{m=-\infty}^{\infty} \exp\left[-\frac{I}{2\hbar^2}\frac{(2\pi m)^2}{\beta}\right],$$

and provide the physical interpretation of the prefactor $\mathcal{Z}_0$.

(d) Making use of the free particle propagator, $\langle q|\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{\hat{p}^2}{2M}t\right]|q\rangle = \left(\frac{M}{2\pi\mathrm{i}\hbar t}\right)^{1/2}$, and the Poisson summation formula,

$$\sum_{m=-\infty}^{\infty} h(m) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi\; h(\phi)\, \mathrm{e}^{2\pi i n\phi},$$

show that this expression is consistent with that obtained in part (a).

(e) Consider now a one-dimensional array of coupled quantum rotors (with unit spacing) defined by the Hamiltonian,

$$\hat{H} = \sum_n \left[\frac{\hat{\ell}_n^2}{2I} - J\cos(\phi_{n+1} - \phi_n)\right],$$

where $J > 0$ represents the strength of the coupling. Taking $\phi_{n+1} - \phi_n \ll 1$, show that the path integral of the one-dimensional system has the continuum limit

$$\mathcal{Z} = \int D\phi(x,\tau) \exp\left[-\frac{1}{\hbar^2}\int_0^\beta d\tau \int dx \left(\frac{I}{2}\dot{\phi}^2 + \frac{J}{2}(\partial_x\phi)^2\right)\right].$$

Comparing this expression to that of a superfluid, discuss the nature of the low energy collective excitations of the array.

4

As a model of itinerant ferromagnetism, consider a variant of the Hubbard Hamiltonian describing the contact interaction of electrons,

$$\hat{H} = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} \, c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - 2U \sum_{\mathbf{n}} \left(\hat{\mathbf{S}}^2_{\mathbf{n}} - \frac{3}{4} \hat{n}_{\mathbf{n}} \right) ,$$

where the operators, $c^{\dagger}_{\mathbf{k}\sigma}$ and $c_{\mathbf{k}\sigma}$, obey fermionic anticommutation relations, $\hat{\mathbf{S}}_{\mathbf{n}} = (1/2) \sum_{\alpha,\beta} c^{\dagger}_{\mathbf{n}\alpha} \sigma_{\alpha\beta} c_{\mathbf{n}\beta}$ denotes the electron spin operator at lattice site $\mathbf{n}$, and $\hat{n}_{\mathbf{n}} = \sum_{\sigma} c^{\dagger}_{\mathbf{n}\sigma} c_{\mathbf{n}\sigma}$ denotes the local number operator. The sum on $\mathbf{n}$ runs over the N sites of a cubic lattice, and the sum on reciprocal lattice vectors, $\mathbf{k}$, spans the first Brillouin zone.

(a) Rewriting the Hamiltonian in normal ordered form, express the quantum partition function, $\mathcal{Z} = \mathrm{tr}\, \mathrm{e}^{-\beta(\hat{H} - \mu \hat{N})}$, in the form of a coherent state field integral, where $\beta = 1/k_{\mathrm{B}}T$ and μ denotes the chemical potential.

(b) By evaluating the field integral, show that the Helmholtz free energy of the non-interacting electron gas ($U = 0$) is given by

$$F = -2k_{\mathrm{B}}T \sum_{\mathbf{k}} \ln[1 + \mathrm{e}^{-\beta(\epsilon_{\mathbf{k}} - \mu)}] .$$

[*Hint: When summed over fermion Matsubara frequencies, you may note that* $\sum_{\omega_n} \ln(-\mathrm{i}\omega_n + z) = \ln(1 + \mathrm{e}^{-\beta z})$.]

(c) Making use of a Hubbard-Stratonovich decoupling scheme, show that the quantum partition function of the interacting Hamiltonian can be cast in the form, $\mathcal{Z} = \int D\mathbf{M} \int D(\bar{\psi}_{\sigma}, \psi_{\sigma}) \, \mathrm{e}^{-S}$, where the action is given by

$$S = \int_0^{\beta} d\tau \sum_{\mathbf{k}\sigma} \bar{\psi}_{\mathbf{k},\sigma} \left(\partial_{\tau} + \epsilon_{\mathbf{k}} - \mu\right) \psi_{\mathbf{k}\sigma} + \int_0^{\beta} d\tau \sum_{\mathbf{n}} \left[\frac{\mathbf{M}^2_{\mathbf{n}}}{2U} - \sum_{\alpha,\beta} \bar{\psi}_{\mathbf{n}\alpha} \, \mathbf{M}_{\mathbf{n}} \cdot \sigma_{\alpha\beta} \, \psi_{\mathbf{n}\beta} \right] .$$

[*Hint: You may wish to consider proving this result by performing the functional integral over* $\mathbf{M}_{\mathbf{n}}(\tau)$ *and comparing your answer with that of part (a).*]

(d) In the mean-field approximation, the Hubbard-Stratonovich field, $\mathbf{M}_{\mathbf{n}}$, can be taken as spatially and temporally homogeneous, oriented along, say, the z-direction, $\mathbf{M}_{\mathbf{n}} = M \, \hat{\mathrm{e}}_z$. In this approximation, integrating out the fermion fields $\bar{\psi}_{\sigma}$ and ψ_{σ}, show that $\mathcal{Z} = \int DM \, \mathrm{e}^{-S_{\mathrm{eff}}}$, where

$$S_{\mathrm{eff}}[M] = -\sum_{\mathbf{k},\sigma} \ln\left[1 + \mathrm{e}^{-\beta(\epsilon_{\mathbf{k}} - \mu - \sigma M)}\right] + \beta N \frac{M^2}{2U} .$$

(e) In the saddle-point (mean-field) approximation, show that

$$M = \frac{U}{2} \sum_{\sigma=\pm 1} \int d\epsilon \, \nu(\epsilon) \, n_{\rm F}(\epsilon - \sigma M) \, \sigma \, ,$$

where $\nu(\epsilon)$ represents the density of states of the non-interacting electron gas at energy ϵ, and $n_{\rm F}(\epsilon) = 1/(1 + {\rm e}^{\beta(\epsilon-\mu)})$ denotes the Fermi distribution function.

(f) Using the expansion, $n_{\rm F}(\epsilon - \sigma M) = n_{\rm F}(\epsilon) - \sigma M \partial_\epsilon n_{\rm F}(\epsilon) + O(M^2)$, show that the system becomes unstable towards the development of a non-zero M when $U > U_c$, where $U_c \nu(0) \simeq 1$. Comment on the expected properties of the low-energy excitations in this ordered phase.

END OF PAPER