UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 4 June, 2014 1:30 pm to 4:30 pm

PAPER 8

ANALYSIS ON POLISH SPACES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What is a *topologically complete* subspace of a metric space?

Show that a topologically complete subspace Y of a complete metric space (X, d) is a G_δ subset.

State whether the converse is true.

Show that a topologically complete normed space $(E, \|.\|)$ is a Banach space.

Show that the open unit ball of the dual of an infinite-dimensional Banach space, with the weak*-topology is not topologically complete.

2

Let $P(X)$ be the set of Borel probability measures on a Polish metric space (X, d). What is the *weak topology* w on $P(X)$? What is the metric β? How are they related?

Suppose that A is an open subset of X. Show that the function $\mu \to \mu(A)$ is lower semi-continuous on $(P(X), w)$. State necessary and sufficient conditions, in terms of open sets, and in terms of closed sets, for a sequence in $P(X)$ to converge in the topology w.

Suppose that $i : X \to \tilde{X}$ is a homeomorphism of X onto a dense subspace $i(X)$ of a compact metric space $(\tilde{X}, \tilde{d})$. If $\mu \in P(X)$, let $j(\mu) = i_*(\mu)$, the push-forward measure of μ. Show that $j : P(X) \to P(\tilde{X})$ is a homeomorphism, when $P(\tilde{X})$ is given its weak topology $\tilde{w}$.

Show that a w-closed uniformly tight subset S of $P(X)$ is w-compact.

Use this to show that $(P(X), w)$ is a Polish space. [You may assume that a β-totally bounded set is uniformly tight.]

3

Suppose that X and Y are Polish spaces, that $\mu \in P(X)$ and $\nu \in P(Y)$, that c is a continuous non-negative cost function on $X \times Y$, and that π is a transport plan. What does it mean to say that π is *c-monotone*, and that π is *strictly c-monotone*?

Show that a c-monotone transport plan is strictly c-monotone, and that it is optimal. State whether the converse is true.

Let $X = [0, 1]$ and $Y = [1, 2]$, with their usual topologies, and suppose that μ and ν have continuous strictly positive densities f and g respectively. What is the set of optimal deterministic transport plans

(a) when $c(x, y) = y - x$, and

(b) when $c(x, y) = (y - x)^2$?

Justify your answers.

4

Suppose that K is a compact convex metrizable set. What is the *upper envelope* $\overline{f}$ of a bounded function f on K?

Show that if μ is a Borel probability measure on K and $f \in C(K)$ then there exists a Borel probability measure ν on K such that $\int_K f\, d\nu = \int_K \overline{f}\, d\mu$ and $\int_K g\, d\nu \leqslant \int_K \overline{g}\, d\mu$ for all $g \in C(K)$.

State and prove Choquet's theorem. [You may assume that $Ex(K)$ is a G_δ set and that there is a strictly convex function in $C(K)$.]

Let f be in the unit ball of $(L^\infty[0, 1], \|.\|_\infty)$. Find a measure ν which satisfies the conclusions of Choquet's theorem. [*Hint: First consider the case where* $f = I_A - I_B$, *where* A *and* B *are disjoint.*]

END OF PAPER