UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 10 June, 2014 9:00 am to 11:00 am

PAPER 79

ADDITIVE COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State and prove Roth's theorem for arithmetic progressions of length 3.

(ii) Let θ be a positive real number. Explain briefly how to generalize Roth's theorem to a statement about subsets $A \subset \mathbb{Z}$ for which there are at least $\theta|A|^3$ quadruples $(a, b, c, d) \in A^4$ with $a + b = c + d$.

2

Proving any results you need along the way, show that there exists a constant C such that for every real number α and every $\epsilon > 0$ there exists a positive integer $n \leqslant \epsilon^{-C}$ such that $\|\alpha n^2\| < \epsilon$. [*Here* $\|x\|$ *denotes the distance from* x *to the nearest integer.*]

3

Let $A \subset \mathbb{Z}$ be a set of size n, let $\theta > 0$, and suppose that there are at least θn^3 quadruples $(a, b, c, d) \in A^4$ such that $a + b = c + d$. Prove that $2A - 2A$ contains an arithmetic progression of length at least n^γ, where γ is a positive constant that depends on θ only. [*Basic facts about Freiman homomorphisms may be assumed, but other results that you use should be proved.*]

4

Give in outline a proof of Szemerédi's theorem for progressions of length 4. [*Your main priority should be to make clear the global structure of the proof. However, you should include proofs of at least some of the steps you judge to be important. For the purposes of this question, results connected with Freiman's theorem should be regarded as background knowledge.*]

END OF PAPER