UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 5 June, 2014 1:30 pm to 4:30 pm

# PAPER 77

# COMPLEX AND BIOLOGICAL FLUIDS

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Consider a straight, rigid rod of length $L$ moving in a Newtonian fluid of dynamic viscosity $\mu$. Due to time-varying external forces and moments, the rod is made to oscillate in position and orientation around the configuration where the rod is aligned with the $y$ direction and its centre is located at the origin. The viscous forces acting on the rod are fully characterized by two resistive drag coefficients, $c_\perp$ and $c_\parallel$. The rod oscillates its centre along the $x$ direction as $x(t) = \epsilon a \cos(\omega t)$, where $a$ has units of length and $\epsilon$ is a dimensionless parameter. It also oscillates its orientation in the $(x, y)$ plane with an angle $\theta = \epsilon \cos(\omega t + \phi)$ about the $y$ direction, where $\phi$ is a constant.

(a) Using symmetry arguments, show that the time-averaged force induced by the rod on the fluid in the $x$ direction is zero. Why can the same arguments not be used to rule out a force in the $y$ direction?

(b) Assuming that $\epsilon \ll 1$, calculate the time-averaged force induced by the rod on the fluid in the $y$ direction at leading order in $\epsilon$.

(c) Which value of the phase $\phi$ leads to the maximum force in the $y$ direction? Use a physical argument to rationalize your result. Why is there no force when $\phi = 0$ or $\pi$?

(d) Compute the time-averaged rate of working of the rod against the fluid at leading order in $\epsilon$. Show in particular that the work associated with the rotational component of the rod's motion is of the same order as the work associated with its translation.

(e) If, instead, the rod is located in a linear viscoelastic fluid, resistive-force theory is modified in the following manner: the hydrodynamic force per unit length acting on the rod follows a Maxwell-like relationship

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \delta \mathbf{f} = -\left[c_\parallel \mathbf{tt} + c_\perp (\mathbf{1} - \mathbf{tt})\right] \cdot \mathbf{u},$$

where $\lambda$ is the relaxation time, $\mathbf{t}$ the tangent to the rod centreline, and $\mathbf{u}$ the velocity of the rod relative to the fluid. Assuming $\epsilon$ to be small, show that the averaged force in the $y$ direction in the same as in the Newtonian case, but that the rate of working is always smaller. Interpret this result physically.

**2**

Many organisms travel through fluid with a suspended solid matrix, for example bacteria in soil or inside biological tissues. The fluid dynamics inside such complex biological media is well described by a modified Stokes equation, where the pressure $p$ and velocity fields $\mathbf{u}$ satisfy

$$-\boldsymbol{\nabla} p + \mu \nabla^2 \mathbf{u} = \mu \alpha^2 \mathbf{u}, \quad \boldsymbol{\nabla} \cdot \mathbf{u} = 0,$$

with $\alpha$ is a positive constant. As a model for a swimmer, consider Taylor's two-dimensional swimming sheet. A two-dimensional, infinite, periodic swimmer deforms its shape as the travelling wave

$$x_S = x, \; y_S = b \sin(kx - \omega t),$$

where $k$, $\omega$ are positive constants, and swims with velocity $-U\mathbf{e}_x$, with $U > 0$. We solve the problem in the frame of the sheet so that the flow at infinity is $+U\mathbf{e}_x$. We further assume that the amplitude of the wave is much less than its wavelength, i.e. if we write $\epsilon = bk$ we consider the limit $\epsilon \ll 1$.

(a) What are the dimensions of $\alpha$?

(b) Obtain the nondimensional governing equations and boundary conditions using $\omega^{-1}$, $k^{-1}$, and $\mu\omega$ as the characteristic time, length, and pressure scales.

(c) Writing $\mathbf{u} = (\partial_y \psi, -\partial_x \psi)$, derive the partial differential equation satisfied by the streamfunction $\psi$. What are the boundary conditions for $\psi$?

(d) Find the solution for the flow at first order in $\epsilon$.

(e) Calculate the swimming speed $U$ of the sheet at order $\epsilon^2$. Does the sheet go faster or slower than in a simple Newtonian fluid? Find a simple physical argument why this would be expected in a complex medium with a background matrix.

(f) Assuming that the stress tensor remains given by $\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\mathbf{e}$, compute the mean rate of working of the sheet against the fluid. How does it compare with the Newtonian case?

**3**

The leading-order far-field flow induced by a swimming cell such as a bacterium is well described by a stresslet,

$$\mathbf{u} = \frac{S}{8\pi\mu}\left[-\frac{(\mathbf{e}\cdot\mathbf{d})}{r^3} + 3\frac{(\mathbf{d}\cdot\mathbf{r})(\mathbf{e}\cdot\mathbf{r})}{r^5}\right]\mathbf{r},$$

where $S$ is a constant, $\mathbf{r}$ is measured from the swimming cell, and $\mathbf{d}$ and $\mathbf{e}$ are unit vectors. We further assume in what follows that $\mathbf{d} = \mathbf{e}$ for each swimming cell.

(a) Explain physically what a stresslet is and how it is derived from the fundamental solution of Stokes flow (the Stokeslet).

(b) If $S > 0$ show that two identical stresslets separated by a vector $\ell\mathbf{e}$ repel each other. Given an initial separation $\ell_0$, compute the solution for $\ell(t)$.

(c) Consider stresslet swimming parallel to and below a free planar surface. The vector $\mathbf{e}$ is thus in the plane of the surface. Explain briefly what the hydrodynamic image system is for the stresslet in this case, and how it affects the swimming trajectory of the stresslet. Show that the stresslet touches the surface in finite time due to hydrodynamic effects.

(d) Consider now two identical stresslets swimming parallel to a free planar surface, separated by a vector $\ell\mathbf{e}$, and both located a distance $h$ below the free surface. Show that when $h \ll \ell$ the time-evolution of $h$ is unchanged but $\ell$ increases more rapidly than in (b).

(e) The rotlet singularity corresponds to a flow field

$$\mathbf{u} = \frac{1}{8\pi\mu}\frac{\mathbf{R}\times\mathbf{r}}{r^3}.$$

Interpret this solution physically and explain why $\mathbf{R} = \mathbf{0}$ for a cell (ignore the effects of gravity).

(f) In addition to the stresslet, the far-field flow created by a flagellated bacterium such as *E. coli* includes a rotlet-dipole, denoted $\mathbf{u}_{RD}$, obtained by taking the derivative of a rotlet along the $\mathbf{e}$ direction. Explain the physical origin of this rotlet-dipole. How are the orientations of $\mathbf{R}$ and $\mathbf{e}$ related? Calculate the solution for $\mathbf{u}_{RD}$.

(g) Consider now the cell swimming parallel to, and below, a free surface at $z = 0$. Let $\mathbf{e}_z$ denote the normal to the surface and $z = -h$ be the location of the rotlet-dipole. Show using symmetry arguments that the rotlet-dipole, on its own, cannot lead to hydrodynamic attraction/repulsion by the surface. Show that the image of the rotlet-dipole leads to a flow with a non-zero $z$-component of vorticity, $\omega_z = \partial_x u_y - \partial_y u_x$, at the location of the cell. How will this vorticity impact the trajectory of the swimming cell?

**4**

Consider an infinite, periodic flagellum swimming as the result of a travelling-wave deformation of its inextensible shape. In the wave frame, the shape of the waveform is prescribed by the functions $(x(s), y(s))$ where $x(s+\Lambda) = x(s) + \lambda$, $y(s+\Lambda) = y(s)$ and $s$ is the arclength along the flagellum. If $V$ is the speed of the travelling wave relative to the flagellum, and assuming that the hydrodynamics are governed by resistive-force theory, show that the flagellum swims at speed $U$ in the direction opposite to the wave with

$$\frac{U}{V} = \frac{(1-\rho)(1-\beta)}{1+\beta(\rho-1)},$$

where $\rho$ is a ratio of drag coefficients and $\beta$ is a geometrical integral of the flagellum's shape which you should derive.

**END OF PAPER**