UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 9:00 am to 11:00 am

PAPER 76

CONVECTION

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the temperature evolution equation for two-dimensional long-wavelength convection without Boussinesq symmetry, where the temperature $\Theta = \Theta(x,t)$:

$$\frac{\partial \Theta}{\partial t} = -\Theta - \mu\Theta_{xx} - \Theta_{xxxx} - s(\Theta_x^2)_x + (\Theta_x^3)_x.$$

Solutions are supposed bounded in $-\infty < x < \infty$.

(i) Defining $\langle \cdot \rangle$ as the usual average in x, show that

$$\frac{1}{2}\frac{d}{dt}\big\langle \Theta^2 \big\rangle = -\big\langle \Theta^2 \big\rangle + \mu\big\langle \Theta_x^2 \big\rangle - \big\langle \Theta_{xx}^2 \big\rangle + s\big\langle \Theta_x^3 \big\rangle - \big\langle \Theta_x^4 \big\rangle.$$

Using the inequality $A^2 + B^2 \geqslant 2AB$, or otherwise, show that

$$\frac{1}{2}\frac{d}{dt}\big\langle \Theta^2 \big\rangle \leqslant \big\langle (\mu - 2)\Theta_x^2 + s\Theta_x^3 - \Theta_x^4 \big\rangle \leqslant \left(\mu - 2 + \frac{s^2}{4}\right)\big\langle \Theta_x^2 \big\rangle$$

Deduce that there can be no growing solutions if $\mu < 2 - s^2/4$.

(ii) Now suppose that $\mu = 2 + \epsilon^2\mu_2 + \ldots$, where $\epsilon \ll 1$, and that $\Theta = \epsilon\Theta_1 + \epsilon^2\Theta_2 + \epsilon^3\Theta_3 + \ldots$. Seek steady solutions for which $\Theta_1 = Ae^{ix} + c.c.$, where A is a complex constant. Show that $|A|^2 = p\mu_2$ at leading order, where $p = p(s)$ is to be determined.

Explain why there is no term $\ldots + \epsilon\mu_1 \ldots$ in the expansion of μ.

2

A layer of fluid heated uniformly from below, close to the critical temperature difference for the onset of convection, has a small travelling temperature perturbation of the form $\epsilon \cos(3k_c(x+\Omega t))$ applied to the lower boundary, where k_c is the critical wavenumber. Assume that the leading order temperature field is proportional to $A(t)\exp(ik_c x) + c.c.$, and use symmetry or physical arguments to argue that if $\epsilon \ll 1$ and the system is very close to critical, then A obeys the approximate equation (where T is some scaled time)

$$A_T = \tilde{\mu}A - c|A|^2 A + \tilde{\epsilon}A^{*2}\exp(3i(\tilde{\Omega}T + \delta))$$

for some real δ, where $\tilde{\epsilon} \propto \epsilon$, $\tilde{\Omega} \propto \Omega$, and $\tilde{\mu}, c$ are real.

(i) Show by writing $A = C\exp(i\tilde{\Omega}T + \delta)$, and rescaling time and the amplitude of C, that the equation can be reduced to a canonical form

$$C_T + i\omega C = \mu C - |C|^2 C + C^{*2}.$$

(ii) By writing $C = Re^{i\theta}$, find equations for the steady states $R = R_0, \theta = \theta_0$ and show that R_0 satisfies $(\mu - R_0^2)^2 + \omega^2 = R_0^2$. Show that there are either two or no non-zero steady states and give a condition on μ for such states to exist. Show that when two non-zero states exist for a given value of μ the one with the larger value of R_0 is stable, while the other is unstable.

(iii) Now consider the situation where $|\mu|, |\omega| \ll 1$. Assuming $\mu = \hat{\mu}\omega^2$, $C = \hat{C}\omega$, $T = \tau/\omega$, derive the reduced system $\hat{C}_\tau + i\hat{C} = \hat{C}^{*2} + \omega(\hat{\mu}\hat{C} - |\hat{C}|^2\hat{C})$. Show that when the small terms in ω are ignored there are an infinite number of periodic solutions for which $H = |\hat{C}|^2 + (\hat{C}^3 - \hat{C}^{*3})/3i$ is constant. Discuss the nature of the dynamics when the small terms in ω are included. [Detailed calculation is not required.]

3

Write an essay about rotating convection. Your essay should cover linear stability theory, including the conditions under which steady and oscillatory convection can occur, dynamics of weakly nonlinear oscillations, wavelength selection at large Taylor number and the Kuppers–Lortz instability.

END OF PAPER