UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 9:00 am to 11:00 am

PAPER 75

SOUND GENERATION AND PROPAGATION

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

For this question, take the equations of mass and momentum conservation to be

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0. \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\Big(\rho u_i u_j + (p+\chi)\delta_{ij} - \sigma_{ij}\Big) = 0, \tag{2}$$

where σ_{ij} is the viscous stress tensor and $-\boldsymbol{\nabla}\chi$ is a potential body force. For part (b), ignoring viscous dissipation and thermal conduction, the energy equation may be taken as

$$\frac{\partial p}{\partial t} + u_j\frac{\partial p}{\partial x_j} = c^2\left(\frac{\partial \rho}{\partial t} + u_j\frac{\partial \rho}{\partial x_j}\right), \qquad \textit{where} \quad c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s. \tag{3}$$

For a perfect gas with ratio of specific heats $\gamma = c_p/c_v$,

$$c^2 = \frac{\gamma p}{\rho} = (\gamma - 1)c_p T.$$

(a) Explain what is meant by an *acoustic analogy.* Let $\hat{\rho}_0(\boldsymbol{x})$ and $\hat{c}_0^2(\boldsymbol{x})$ be arbitrary functions of position with no time dependence, and let $\hat{p}_0 = P - \chi$ with P an arbitrary constant. Using only conservation of mass (1) and momentum (2), derive the acoustic analogy

$$\frac{1}{\hat{c}_0^2}\frac{\partial^2}{\partial t^2}(p - \hat{p}_0) - \nabla^2(p - \hat{p}_0) = \frac{\partial^2 W_{ij}}{\partial x_i x_j} + \frac{\partial^2 Q}{\partial t^2}, \tag{4}$$

where

$$W_{ij} = \rho u_i u_j - \sigma_{ij}, \qquad \text{and} \qquad Q = \frac{1}{\hat{c}_0^2}(p - \hat{p}_0) - (\rho - \hat{\rho}_0).$$

Explain how the arbitrary functions $\hat{c}_0^2$ and $\hat{\rho}_0$ might sensibly be chosen when attempting to predict the sound generated by a region of flow surrounded by stationary fluid.

(b) Now consider small perturbations $\big(\boldsymbol{u'}(\boldsymbol{x},t),\, p'(\boldsymbol{x},t),\, \rho'(\boldsymbol{x},t)\big)$ to a static fluid $\big(\mathbf{0},\, p_0(\boldsymbol{x}),\, \rho_0(\boldsymbol{x})\big)$, and neglect viscosity and thermal diffusivity. Show that the static fluid satisfies the governing equations (1–3) provided $\boldsymbol{\nabla}(p_0 + \chi) = 0$. Neglecting quantities quadratic or smaller in the small perturbation, derive from the governing equations (1–3) the "wave" equation

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \frac{1}{c_0^2 \rho_0}\boldsymbol{\nabla} p_0 \cdot \boldsymbol{\nabla} p' - \frac{1}{\rho_0}\boldsymbol{\nabla}\rho_0 \cdot \boldsymbol{\nabla} p',$$

and hence show that, for a perfect gas,

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\rho_0}{p_0^{1/\gamma}}\boldsymbol{\nabla}\cdot\left(\frac{p_0^{1/\gamma}}{\rho_0}\boldsymbol{\nabla} p'\right) = 0. \tag{5}$$

By comparing (5) and (4) without performing further calculations, briefly justify why the $\partial^2 Q/\partial t^2$ term in (4) cannot be interpreted unambiguously as a noise source.

(c) The Greens' function $G(\boldsymbol{x},t;\ \boldsymbol{y},\tau)$ for (5) satisfies

$$\frac{1}{c_0^2}\frac{\partial^2 G}{\partial t^2} - \frac{\rho_0}{p_0^{1/\gamma}}\boldsymbol{\nabla}\cdot\left(\frac{p_0^{1/\gamma}}{\rho_0}\boldsymbol{\nabla} G\right) = \delta(\boldsymbol{x}-\boldsymbol{y})\delta(t-\tau).$$

Assuming suitable boundary conditions, show that G satisfies the reciprocity condition

$$G(\boldsymbol{y}_1,t;\ \boldsymbol{y}_2,\tau_2) = G(\boldsymbol{y}_2,t+\tau_1-\tau_2;\ \boldsymbol{y}_1,\tau_1)\frac{\left[p_0(\boldsymbol{y}_2)\right]^{1/\gamma}\rho_0(\boldsymbol{y}_1)}{\left[p_0(\boldsymbol{y}_1)\right]^{1/\gamma}\rho_0(\boldsymbol{y}_2)}.$$

[Hint: Fourier transform in time and then consider $\boldsymbol{\nabla}\cdot(fG_1\boldsymbol{\nabla}G_2 - fG_2\boldsymbol{\nabla}G_1)$ *with suitably chosen functions* G_1*,* G_2 *and* f*.]*

2

A semi-infinite 2D waveguide is formed from two rigid plates located at $y=\pm b$ for $x>0$. An incident plane wave propagates in the negative x direction inside the waveguide, with density perturbation

$$\rho_{\text{inc}} = \exp\{\mathrm{i}\omega t + \mathrm{i}k_0 x\}H(b-|y|),$$

where H is the Heaviside step function and $k_0=\omega/c_0$ with c_0 the speed of sound. By writing $\rho=\rho_{\text{inc}}+\phi$ and noting the symmetry in the y-direction, show that the Wiener–Hopf equation for this situation is

$$\frac{1}{L(k)}\left.\frac{\partial\Phi^-}{\partial y}\right|_{y=b} + \left[\Phi^+\right]_{b-}^{b+} = \frac{\mathrm{i}}{k+k_0},$$

where $L(k)=\gamma\sinh(\gamma b)\mathrm{e}^{-\gamma b}$, $\gamma(k)=\sqrt{k^2-k_0^2}$, and $\Phi=\Phi^+ + \Phi^-$ is the x-Fourier transform of ϕ. How should the branch cuts of $\gamma(k)$ be taken?

Solve this Wiener–Hopf equation by assuming the appropriate entire function $E(k)\equiv 0$ to find, for $|y|<b$,

$$\Phi = \frac{\mathrm{i}L^+(-k_0)L^-(k)\cosh(\gamma y)}{(k+k_0)\gamma\sinh(\gamma b)}.$$

What is Φ for $|y|>b$?

By noting that Φ is an even function of γ for $|y|<b$, deduce that the branch cut in Φ in the lower half plane is removable for $|y|<b$. By considering the inverse Fourier transform and deforming the contour of integration into the lower half k-plane, show that ϕ within the waveguide ($x>0$ and $|y|<b$) is given as a sum of waveguide modes propagating in the positive x direction, and find the amplitude of the plane wave mode.

3

Burgers' equation is

$$\frac{\partial f}{\partial Z} - f\frac{\partial f}{\partial \theta} = \alpha\frac{\partial^2 f}{\partial \theta^2}.$$

The inviscid Burgers' equation is obtained by setting $\alpha = 0$.

(a) Show that the inviscid Burgers' equation with initial conditions $f(0,\theta) = f_0(\theta)$ has solution $f(Z,\,\theta_0 - f_0(\theta_0)Z) = f_0(\theta_0)$. Show also that if there is a weak shock at $\theta_s(Z)$ then

$$\frac{\mathrm{d}\theta_s}{\mathrm{d}Z} = -\frac{1}{2}\Big[f(Z,\theta_s+) + f(Z,\theta_s-)\Big].$$

Solve the inviscid Burgers' equation with initial conditions representing a periodic backward-sawtooth wave $f_0(\theta)$, given by

$$f_0(\theta+2) = f_0(\theta) \qquad \text{and} \qquad f_0(\theta) = \theta \qquad \text{for } -1 < \theta < 1,$$

being careful to distinguish between $0 < Z < 1$ and $Z \geqslant 1$. Sketch $f(Z,\theta)$ for $Z = 1/3$ and $Z = 3$.

[Hint: it may help to sketch the characteristics first; when doing so, think of $f_0(\theta)$ as being continuous but very steep at, for example, $\theta = \pm 1$.]

(b) For $\alpha \neq 0$, show that the Cole–Hopf transformation

$$f = 2\alpha\frac{\partial}{\partial\theta}\log\psi$$

can be used to solve Burgers' equation when ψ satisfies a diffusion equation. Given that the general solution to the diffusion equation is

$$\psi(Z,\theta) = \frac{1}{\sqrt{4\pi\alpha Z}}\int_{-\infty}^{\infty}\psi(0,\phi)\exp\left\{-\frac{(\phi-\theta)^2}{4\alpha Z}\right\}\mathrm{d}\phi,$$

show that the solution to the full Burgers' equation with initial conditions given by an N-wave,

$$f_0(\theta) = \begin{cases} -U\theta & |\theta| < L \\ 0 & |\theta| > L \end{cases}$$

may be written as $f(Z,\theta) = 2\alpha\frac{\partial}{\partial\theta}\log\psi$, with

$$\psi(Z,\theta) = \big(1 - I_\alpha(\theta,L,Z)\big) + I_\alpha\big(\theta,\,L(1+UZ),\,Z(1+UZ)\big)\hat{\psi}(Z,\theta)$$

where

$$I_\alpha(\theta,b,w) = \frac{1}{\sqrt{4\pi\alpha w}}\int_{-b}^{b}\exp\left\{-\frac{(\phi-\theta)^2}{4\alpha w}\right\}\mathrm{d}\phi,$$

and

$$\hat{\psi} = \frac{1}{\sqrt{1+UZ}}\exp\left\{\frac{U}{4\alpha}\left(L^2 - \frac{\theta^2}{1+UZ}\right)\right\}.$$

Now consider the limit $\alpha \to 0$. In this limit, show that $I_\alpha(\theta,L,w) \approx H(L^2-\theta^2)$, where H is the Heaviside step function. Hence, in this limit, show that $f(Z,\theta) \approx 0$ for $|\theta| \gg L(1+UZ)$, and find an approximation for $f(Z,\theta)$ when $|\theta| \ll L$.

END OF PAPER