UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 2 June, 2014 1:30 pm to 4:30 pm

PAPER 74

PERTURBATION AND STABILITY METHODS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) In the limit $\varepsilon \to 0$ find the leading-order approximations for the roots of the equation

$$\varepsilon\lambda^3 + \lambda^2 + 2\lambda + \varepsilon = 0\,.$$

Distinguish between regular and singular roots.

Use *your results* to obtain an asymptotic solution to the problem

$$\varepsilon y''' + y'' + 2y' + \varepsilon y = 0\,, \quad x \geqslant 0\,,$$

with

$$y(0) = y'(0) = 0 \quad \text{and} \quad y''(0) = 1\,.$$

Estimate the error in the result for $y(x)$ and sketch $y(x)$.

(b) The Bessel function $J_n(x)$ is defined for real x and integer $n \geqslant 0$ as

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)d\theta\,.$$

Find leading-order asymptotic expressions for

(i) $J_n(x)$ for $x \to +\infty$ and n fixed.

(ii) $J_n(n\sec\alpha)$ for $n \to \infty$ and α fixed ($\alpha > 0$).

(iii) $J_n(n)$ for $n \to \infty$.

Standard results may be stated without proof. Recall that

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt\,.$$

2

(a) A weakly perturbed harmonic oscillator satisfies the equation

$$\frac{d^2y}{dt^2} + y = \varepsilon f\left(y, \frac{dy}{dt}\right).$$

In the limit $\varepsilon \to 0$, use the method of multiple scales to find equations for the slow evolution of the amplitude R and phase θ of the oscillations in terms of averages $\langle f \cos(t+\theta)\rangle$ and $\langle f \sin(t+\theta)\rangle$ to be specified.

(i) What further may be deduced if f depends only on y? Find R and θ explicitly in the case

$$f = y^3.$$

(ii) What further may be deduced if f depends only on dy/dt? Find R and θ explicitly in the case

$$f = \left(\frac{dy}{dt}\right)^3.$$

Recall that $\cos^4\alpha = \frac{1}{8}\cos 4\alpha + \frac{1}{2}\cos 2\alpha + \frac{3}{8}$.

(b) If $m \to 1$ with $m < 1$ find a leading-order asymptotic approximation for the elliptic function

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - m\sin^2\theta)^{1/2}}.$$

What is the order of the next term?

3

(a) The function $y(x)$ satisfies the differential equation

$$(1+\varepsilon)x^2y' = (1-\varepsilon)\varepsilon\, xy^2 - (1+\varepsilon)\varepsilon\, x + \varepsilon\, y^3 + 2\varepsilon^2\, y^2 \quad \text{in} \quad 0 \leqslant x \leqslant 1\,,$$

where $0 < \varepsilon \ll 1$. If $y(1) = 1$, calculate three terms of the outer solution of y. Locate the non-uniformity of the asymptoticness, and hence the rescaling for an inner region. Thence find two terms for the inner solution.

[*Hint: The general solution to*

$$\xi^2 g' - \left(\frac{3\xi}{2+\xi}\right) g = -\left(\frac{\xi}{2+\xi}\right)^{\frac{3}{2}}$$

is

$$g(\xi) = \frac{(1+k\xi)\xi^{\frac{1}{2}}}{(2+\xi)^{\frac{3}{2}}}\,,$$

for some constant k.]

(b) Using matched asymptotic expansions find the value of $z'(0)$ to leading order if $z(1) = e^{-1}$ and $z(x)$ satisfies the equation

$$(x - \varepsilon z)z' + xz = e^{-x}\,.$$

4

The classical unsteady boundary-layer equations are

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y = \mathcal{U}_t + \mathcal{U}\mathcal{U}_x + \bar{u}_{yy}\,, \quad \bar{u}_x + \bar{v}_y = 0\,,$$

where x, y are Cartesian co-ordinates, t is time, $\bar{u} \equiv \bar{u}(x, y, t)$ and $\bar{v} \equiv \bar{v}(x, y, t)$ are velocity components, and $\mathcal{U}(x, t)$ is the 'slip' velocity. Appropriate boundary conditions are

$$\bar{u} = \bar{v} = 0 \quad \text{on} \quad y = 0\,, \quad \text{and} \quad \bar{u} \to \mathcal{U}(x, t) \quad \text{as} \quad y \to \infty\,.$$

Consider the *linear* instability of a solution $(\bar{u}, \bar{v}) = (U, V)$ of the boundary-layer equations by writing

$$(\bar{u}, \bar{v}) = (U, V) + \delta(\tilde{u}, \tilde{v})\,,$$

where $0 < \delta \ll 1$. Find the linear equations that $(\tilde{u}, \tilde{v})$ satisfy, and state appropriate boundary conditions for $(\tilde{u}, \tilde{v})$.

On the basis of the simplifying assumption $U \equiv U(y)$, $V = 0$, explain why it is possible to seek a normal mode solution of the form

$$(\tilde{u}, \tilde{v}) = (u(y),\, v(y)) \exp(i\alpha(x - ct))\,.$$

Derive a governing equation and boundary conditions for v. Explain why, if c is the eigenvalue for wavenumber α, the complex conjugate c^* is the eigenvalue for wavenumber $-\alpha$. Henceforth take $\alpha > 0$.

Suppose that the unperturbed velocity profile U, in addition to satisfying the no-slip condition $U(0) = 0$ and the free-stream condition $U \to \mathcal{U}$ as $y \to \infty$, has a point of zero shear at $y = y_c > 0$, i.e. $U_y(y_c) = 0$. In particular, assume that for $|y - y_c| \ll 1$ the velocity profile expands as

$$U = U(y_c) + \tfrac{1}{2}(y - y_c)^2 + \dots\,.$$

Next, assume that the wavenumber is real and large in magnitude, i.e.

$$\alpha = \frac{k}{\varepsilon} \quad \text{where} \quad 0 < \varepsilon \ll 1\,,$$

and seek an asymptotic solution for v by expanding v and c as

$$(v, c) = (v_0, c_0) + \varepsilon^{\frac{1}{2}}(v_1, c_1) + \dots\,.$$

Discuss the validity of the leading-order outer solution

$$v_0 = \begin{cases} U - c_0 & \text{for} \quad y > y_c \\ 0 & \text{for} \quad y < y_c \end{cases}\,,$$

where you should make an appropriate choice for c_0.

Explain why a 'critical layer' exists close to $y = y_c$, and derive appropriate inner scalings

$$y - y_c = \varepsilon^p Y\,, \quad v = \varepsilon^q w(Y) + \dots\,,$$

where p and q are to be determined. Show that

$$kYw - k(\tfrac{1}{2}Y^2 - c_1)w_Y = iw_{YYY}\,.$$

Having found a solution for v_1, give matching conditions for w.

Find a transformation of variables $(Y, c_1, w) \to (z, C, W)$ that reduces the eigenvalue problem to

$$W_{zzz} - (\tfrac{1}{2}z^2 - C)W_z + zW = 0\,,$$
$$W \to \pm(\tfrac{1}{2}z^2 - C) \quad \text{as} \quad z \to \pm\infty\,.$$

Given that the eigenvalues are

$$C = \frac{4n+7}{\sqrt{2}} \quad \text{for} \quad n = -2, -1, 1, 2, \ldots\,,$$

comment on the stability of the flow.

END OF PAPER