UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 1:30 pm to 4:30 pm

PAPER 73

SLOW VISCOUS FLOW

*Attempt **QUESTION 1** and no more than **TWO OTHER** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

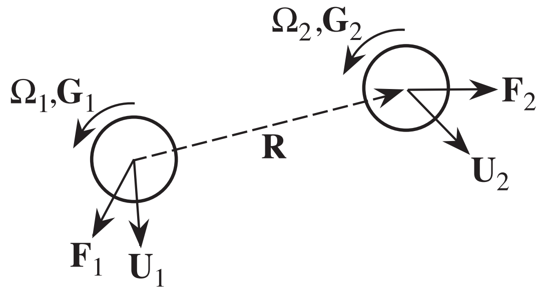
You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State the Papkovich–Neuber representation for the velocity and pressure in Stokes flow. Use this representation, explaining your choice of trial harmonic potentials, to determine the velocity field due to a rigid sphere of radius a moving with velocity $\mathbf{U} + \mathbf{\Omega} \wedge \mathbf{x}$ through unbounded fluid of viscosity μ that is otherwise at rest.

[*You may assume below that the force and couple required on this sphere are* $6\pi\mu a\mathbf{U}$ *and* $8\pi\mu a^3\mathbf{\Omega}$, *respectively.*]

(b) Two identical rigid spheres of radius a, denoted by $i = 1, 2$, undergo rigid-body motions $\mathbf{U}_i + \mathbf{\Omega}_i \wedge \mathbf{x}$ in unbounded fluid under the influence of applied forces $\mathbf{F}_i$ and couples $\mathbf{G}_i$. As shown in the diagram, the $\mathbf{U}_i$ and $\mathbf{F}_i$ are coplanar and perpendicular to the $\mathbf{\Omega}_i$ and $\mathbf{G}_i$, and $\mathbf{R}$ is the vector distance from sphere 1 to sphere 2.



(i) If $\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{0}$ show that $R = |\mathbf{R}|$ is constant. If, in addition, $\mathbf{G}_1 = \mathbf{G}_2$, what can be said about the $\mathbf{U}_i$ and $\mathbf{\Omega}_i$?

(ii) State the minimum dissipation theorem. If $\mathbf{G}_1 = \mathbf{G}_2 = \mathbf{F}_2 = \mathbf{0}$, show that $\mathbf{U}_1 \cdot \mathbf{F}_1 \leqslant F_1^2/(6\pi\mu a)$? Justify your answer carefully. Does this inequality hold if $\mathbf{G}_1 \neq \mathbf{0}$? Why, or why not?

You are now given that $\mathbf{F}_1 = 6\pi\mu a\mathbf{V}$, where $\mathbf{V}$ is a constant, $\mathbf{G}_1 = \mathbf{G}_2 = \mathbf{0}$, and $R \gg a$.

(iii) For $\mathbf{F}_2 = 6\pi\mu a\mathbf{V}$, find the $\mathbf{U}_i$ and $\mathbf{\Omega}_i$ correct to $O(a^2/R^2)$.

(iv) For $\mathbf{U}_2 = \mathbf{V}$, find $\mathbf{F}_2$ and $\mathbf{U}_1$ correct to $O(a^2/R^2)$.

(v) For $\mathbf{F}_2 = \mathbf{0}$, find the order of magnitude of $\mathbf{U}_1 - \mathbf{V}$ and of $\mathbf{\Omega}_1$.

[*You may assume the Faxén formulae*

$$\mathbf{U} = \frac{\mathbf{F}}{6\pi\mu a} + \mathbf{u}_\infty + \frac{a^2}{6}\nabla^2\mathbf{u}_\infty \ , \quad \mathbf{\Omega} = \frac{\mathbf{G}}{8\pi\mu a^3} + \frac{1}{2}\boldsymbol{\omega}_\infty \ ,$$

but should explain how you apply them.]

2

A rigid horizontal boundary is covered with a thin liquid film of height $h(x)$, viscosity μ and density ρ, where x is the horizontal coordinate. Insoluble surfactant with concentration $C(x)$ and diffusivity D_s resides on the surface of the film and reduces its surface tension according to the equation $\gamma(C) = \gamma_0 - AC$, where γ_0 and A are positive constants. The flow is steady.

Assume that the variations of h and C are such that lubrication theory applies. Hence find the (dimensional) horizontal fluxes q and j of liquid and surfactant produced by the combined effects of Marangoni stresses, gravity, capillarity and surfactant diffusion. Let dimensionless variables $\Gamma = C/C_0$ and $X = x/L$ be defined, where C_0 and L are constants. Show that the corresponding dimensionless fluxes can be written as

$$Q = -\tfrac{1}{2}H^2\Gamma_X - \tfrac{1}{3}H^3\left(H - G\{(1-\alpha\Gamma)H_{XX}\}\right)_X, \tag{1}$$

$$J = -\left(H\Gamma + \Delta\right)\Gamma_X - \tfrac{1}{2}H^2\Gamma\left(H - G\{(1-\alpha\Gamma)H_{XX}\}\right)_X, \tag{2}$$

where Q and J, the dimensionless height H, and the dimensionless parameters Δ, G and α should all be defined in terms of the dimensional quantities.

Assume that Δ and G are negligible from now on. Show that (1) and (2) can be written in the form

$$\Gamma_X = f(\Gamma, H), \qquad HH_X = g(\Gamma, H)$$

for some functions f and g to be determined. Sketch the trajectories in the (Γ, H) phase plane, assuming that Q and J are strictly positive. [*Hint: Consider the signs of H_X and Γ_X in the quarter-plane $H \geqslant 0, \Gamma \geqslant 0$.*]

Now consider a finite-length film in $0 \leqslant X \leqslant 1$ with imposed boundary conditions $\Gamma(0) = 1$, $H(0) = H_0$ and $\Gamma(1) = 1 - \delta$, where $0 \leqslant \delta \leqslant 1$.

(i) If the surfactant flux is zero, find solutions for $H(X)$, $\Gamma(X)$ and then Q. [The equations for H and Γ can involve Q.] Sketch the shape of the film, showing the velocity profile in the liquid.

(ii) If the liquid flux is zero, find an equation for Γ as a function of H, and hence an implicit equation in the form $X = F(H)$ for the height of the film.

Consider the case $H_0 = 0$. Sketch the shape of the film, showing the velocity profile in the liquid. Show that

$$J = \frac{\delta^{3/2}(1 - \frac{3}{5}\delta)}{2\sqrt{3}}.$$

Sketch the variation of J with δ for $0 \leqslant \delta \leqslant 1$, and comment briefly on why the location of the maximum J may or may not be surprising physically.

3

A long cylindrical tube of radius R, with its axis vertical, is filled with fluid of density ρ_0 and viscosity μ. A source of light fluid of density $\rho_0 - \tilde{\rho}$ and much smaller viscosity $\lambda\mu$ (i.e. $\tilde{\rho} > 0$ and $\lambda \ll 1$) produces a thin axisymmetric plume of radius $a(z,t) \ll R$ which rises under its buoyancy along the axis of the tube. In both fluids the modified pressure P is defined by $P = p + \rho_0 g z$.

Write down the governing equations and boundary conditions for the vertical velocity $w(r)$ in the case of steady flow with $a = a_0$, where a_0 is a constant. If there is no pressure gradient $\mathrm{d}P/\mathrm{d}z$ show that

$$w(0)/w(a_0) = 1 + [2\lambda \ln(R/a_0)]^{-1}.$$

[*Note that* $\nabla^2\phi(r,z) = \dfrac{1}{r}\dfrac{\partial}{\partial r} r \dfrac{\partial \phi}{\partial r} + \dfrac{\partial^2 \phi}{\partial z^2}$ *in cylindrical polar coordinates.*]

Now consider the case of varying $a(z,t)$ with $|\partial a/\partial z| \ll 1$. Assume that $(a/R)^4 \ll \lambda \ll [\ln(R/a)]^{-1}$ and that there is no net flux along the tube. Denote the modified pressures inside and outside the plume by P_i and P_o, respectively. Show that the vertical flux q inside the plume is given at leading order by

$$q = \frac{\pi a^4}{8\lambda\mu}\left(\tilde{\rho} g - \frac{\partial P_i}{\partial z}\right)$$

commenting briefly on any approximations made. [You do not need to determine the external flow.] Show also that $|\partial P_o/\partial z| \ll |\tilde{\rho} g - \partial P_i/\partial z|$. Assume from now on that P_o is negligible.

By approximating the local flow in $a < r \ll R$ as a radial source flow in two dimensions, establish a linear relationship between P_i and $\partial a/\partial t$. Hence derive the dimensionless evolution equation for the propagation of disturbances on the plume

$$\frac{\partial A}{\partial T} + \frac{\partial}{\partial Z}\left(A^2\left[1 - \frac{\partial}{\partial Z}\left(\frac{1}{A}\frac{\partial A}{\partial T}\right)\right]\right) = 0,$$

where $A = a^2/a_0^2$, a_0 is a typical radial scale, and Z and T should be defined.

Find the dispersion relationship for a small perturbation proportional to $\exp[\mathrm{i}(kZ - \omega T)]$ to a uniform plume with $A = 1$. Which way does the perturbation propagate?

Look for travelling-wave solutions of the form $A = f(Z - cT)$, where $f(\zeta) \to f_0$ as $\zeta \to \pm\infty$. Show that such travelling waves satisfy

$$\frac{c}{2}\frac{f'^2}{f^2} + V(f) = \text{const.}$$

where V is to be found. Deduce that the speed is related to the amplitude $f_{\max} = \alpha f_0$ by

$$c\left(1 - \frac{2}{\alpha} + \frac{1}{\alpha^2}\right) = f_0\left(2\ln\alpha - 1 + \frac{1}{\alpha^2}\right).$$

4

A vertical cylinder of radius a is coated with a thin film of fluid of density ρ, viscosity μ and thickness $\epsilon\bar{h}(\bar{z},\bar{t})$, where $\epsilon \ll 1$ and $\bar{z}$ measures the (dimensional) vertical distance down the cylinder. The free surface of the film is acted on by uniform surface tension γ. Use lubrication theory to derive the dimensionless evolution equation

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial z}\left[h^3\Big(\frac{\partial^3 h}{\partial z^3} + \frac{\partial h}{\partial z} + G\Big)\right] = 0, \qquad (1)$$

where $h = \bar{h}/a$ and the other dimensionless variables should be defined.

By setting $h = 1 + \alpha\exp[\mathrm{i}kz + st]$, where $|\alpha| \ll 1$, determine s as a function of k, and describe briefly the evolution of small disturbances to a uniform layer.

Look for solitary travelling-wave solutions to (1) of the form $h(z,t) = H(x)$, where $x = z - ct$ and $H \to 1$ as $x \to \pm\infty$. Find a third-order differential equation for $H(x)$. Now find, as follows, an approximate description of a solitary large-amplitude wave that travels down the cylinder with dimensionless speed $c \gg 1$. Assume that

(i) for $0 < x < 2\pi$, $H(x) = c^{2/3}H_2(x) + H_0(x) + O(c^{-1/3})$;

(ii) near $x = 0$ and $x = 2\pi$ there are short transition regions where the curvature changes rapidly to match the wave to uniform films ahead and behind;

(iii) $G = O(1)$.

Show that the leading-order equation for $0 < x < 2\pi$ is $H_2''' + H_2' = 0$. Determine $H_2(x)$ to within a multiplicative constant using the fact that $H, H' \ll c^{2/3}$ near $x = 0, 2\pi$.

Show that it is possible to define a coordinate X in each of the transition regions in such a way that the leading-order differential equation becomes $H^3 H_{XXX} = H - 1$. Explain why this equation has, to within translations in X, a unique solution $H_-(X)$ with $H_- \to 1$ as $X \to -\infty$ and a one-parameter set of solutions $H_+(X)$ with $H_+ \to 1$ as $X \to +\infty$.

If $H_- \sim \frac{1}{2}\kappa X^2 + \beta_-$ as $X \to \infty$ (with $\kappa = 0.643, \beta_- = 2.90$), show that the maximum (dimensional) film thickness is $2\kappa a\epsilon(3c)^{2/3}$. Sketch $H(x)$ indicating the location of any capillary waves, and show that these waves have (dimensional) wavelength $4\pi a/(3^{5/6}c^{1/3})$.

Assuming that $H_0' = 0$ at $x = 0, 2\pi$, show that

$$H_0(x) = A_0 + B_0\cos x + G(\sin x - x)$$

for some constants A_0 and B_0. There is a unique solution $H_+(X)$ with $H_+'' \to \kappa$ as $X \to -\infty$, and this has $H_+ \sim \frac{1}{2}\kappa X^2 + \beta_+$ (with $\beta_+ = -0.85$). By matching heights near $x = 0, 2\pi$, deduce that the (dimensional) uniform film thickness far from the wave is given by

$$\epsilon a = \frac{2\pi}{\beta_- - \beta_+}\frac{\rho g a^3}{\gamma}.$$

END OF PAPER