UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 4 June, 2014 1:30 pm to 3:30 pm

PAPER 72

DIRECT AND INVERSE SCATTERING OF WAVES

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider an acoustic field $\psi(\mathbf{r})$ generated by an incident wave $\psi_i(\mathbf{r})$ propagating in a medium with refractive index $n(\mathbf{r})$.

(a) Derive the 1st order Rytov approximation for the field, and state under which conditions it is valid.

(b) Show the relation between the 1st order Rytov approximation, $\psi_1^{(R)}(\mathbf{r})$, and the 1st order Born approximation

$$\psi_1^{(B)} = \psi_i(\mathbf{r}) - \int G(\mathbf{r}-\mathbf{r}')[V(\mathbf{r}')\psi_i(\mathbf{r}')]d\mathbf{r}' \ , \tag{1}$$

where $G(\mathbf{r},\mathbf{r}')$ is the free space Green's function, and $V(\mathbf{r}) = [n^2(\mathbf{r}) - 1]$.

(c) Now, assume that the field propagates in a medium with randomly varying refractive index $n(\mathbf{r})$, which is statistically stationary in space and such that $< n(\mathbf{r}) >= 0$.

Denote by W the fluctuating part of V: $W(\mathbf{r}) = V(\mathbf{r}) - < V(\mathbf{r}) >$.

By writing the 1st order Rytov approximation, $\psi_1^{(R)}(\mathbf{r})$, as

$$\psi_1^{(R)}(\mathbf{r}) = A(\mathbf{r})e^{i[\phi_0+\varphi(\mathbf{r})]} \ , \tag{2}$$

where ϕ_0 is deterministic, and decomposing the term $G(\mathbf{r},\mathbf{r}')\psi_i(\mathbf{r}')/\psi_i(\mathbf{r})$ into its real and imaginary parts:

$$G(\mathbf{r},\mathbf{r}')\frac{\psi_i(\mathbf{r}')}{\psi_i(\mathbf{r})} = a(\mathbf{r},\mathbf{r}') + ib(\mathbf{r},\mathbf{r}') \ , \tag{3}$$

write an integral expression for the fluctuating phase $\varphi(\mathbf{r})$.

Hence

(i) derive $< \varphi >$,

(ii) write an expression for $< \varphi^2 >$ in terms of the autocorrelation of the medium.

2

Consider the 2-dimensional problem of a time-harmonic, monochromatic acoustic plane wave ψ_i incident upon a perfectly reflecting randomly rough surface defined by $z = h(x)$, with mean $< h(x) >= 0$ and r.m.s. height σ.

(a) Assuming that the variation in surface height is small, $|kh(x)| \ll 1$, use first order perturbation theory to derive an expression for the scattered field $\psi_s(x, z)$ in the case of Dirichlet boundary conditions: $\psi = 0$ at the surface.

(b) Assuming that the surface is statistically stationary, consider the particular case of a plane wave $\psi_i = e^{ik(x \sin\theta - z \cos\theta)}$, and calculate the mean scattered field $< \psi_s(x, z) >$ in first order perturbation theory.

(c) Still under the assumption of small surface height, and for Dirichlet boundary conditions, use now second order perturbation theory to derive an expression for the scattered field at the surface in the case of a plane wave at normal incidence to the mean plane, $\psi_i = e^{-ikz}$.

(d) Derive an expression for the mean scattered field $< \psi_s(x, 0) >$ at $z = 0$ in second order perturbation theory, and comment on the differences with first order perturbation theory.

3

Consider the inverse problem $Ax = y$, where x is the unknown and A is a compact linear operator between two Hilbert spaces: $A : X \mapsto Y$.

(a) Define the Moore–Penrose generalised inverse $A^\dagger$ and state the *normal equation* satisfied by the generalised solution $x^\dagger$.

(b) Define the set $\{\sigma_n; v_n; u_n\}$, with $n \in \mathbb{N}$, which forms a singular value system for A.

(c) Find a singular value system for the integral operator $A : L^2[0, 2\pi] \mapsto L^2[0, 2\pi]$ defined by:

$$Af(x) := \int_0^{2\pi} K(x - x')f(x')dx' \ , \qquad \begin{array}{ll} x & \in [0, 2\pi] \\ f(x) & \in L^2[0, 2\pi] \end{array} \tag{1}$$

where the kernel $K : \mathbb{R} \mapsto \mathbb{C}$ is a continuously differentiable function, periodic with period 2π, and with Fourier coefficients given by

$$c_n := \int_0^{2\pi} e^{-inx}K(x)dx \neq 0 \ \ , \qquad n = 0, \pm 1, \pm 2, ... \tag{2}$$

[*Hint: You may wish to make use of the orthonormal basis for* $L^2[0, 2\pi]$ *given by* $e_n = \frac{1}{\sqrt{2\pi}}e^{inx}$, $n = 0, \pm 1, \pm 2, ...$]

(d) Use the singular value system found for A in (c), to write an expression for the Moore–Penrose generalised inverse for the inverse problem defined by

$$Af(x) = y \ , \tag{3}$$

where $f(x)$ is unknown, and y is known and given by $y = e^{\alpha x}$, where α is a constant.

END OF PAPER