UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 29 May, 2014 1:30 pm to 4:30 pm

PAPER 70

FLUID DYNAMICS OF THE ENVIRONMENT

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider small-amplitude, two-dimensional monochromatic internal waves in a Boussinesq fluid with constant buoyancy frequency, N^2, where

$$N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz},$$

and ρ_0 is a characteristic value of the background density distribution $\overline{\rho}(z)$. In a stationary ambient fluid, derive the dispersion relation for waves associated with a two-dimensional vertical displacement perturbation

$$\zeta = A_\zeta \cos(kx + mz - \Omega t),$$

where A_ζ, k, m and Ω are all real. Show that statically unstable regions are predicted to develop in the flow if $|A_\zeta| > |1/m|$.

Now consider a situation where the ambient fluid velocity varies with height, such that $\overline{U}(z) = sz$ for $z > 0$ where s is a positive real constant. A two-dimensional wave packet propagates upwards into the upper half-plane, with horizontal wavenumber $k > 0$ and vertical wavenumber $m_0 < 0$ at $z = 0$. Using the ray-tracing equations (which you may quote without proof), show that the intrinsic frequency ω and the horizontal phase speed $c_x = \omega/k$ measured by a stationary observer remain constant for all time. Hence derive an expression for the height z_c of the critical level where $c_x = \overline{U}(z_c)$.

Furthermore, show that the ray followed by this wave packet is defined implicitly as the solution to the differential equation

$$\frac{dx}{dz} = \tan|\Theta| + \frac{ksz}{N\sin|\Theta|\cos^2|\Theta|}, \quad \tan^2\Theta(z) = \frac{N^2}{(\omega - ksz)^2} - 1.$$

You are given that

$$c_{gz}\frac{N^2 A_\zeta^2}{\omega - k\overline{U}} = B,$$

where A_ζ is the displacement amplitude, c_{gz} is the vertical component of the group velocity, and B is a constant. Hence establish that the wave packet is expected to induce statically unstable regions at a height z where

$$\cot|\Theta(z)| = \mathcal{O}([\omega - ksz]^{1/4}).$$

2

You are given that a turbulent buoyant plume in an unstratified environment of constant density ρ_a satisfies the equations

$$\begin{aligned} \frac{d}{dz}Q &= 2\alpha M^{1/2}; \ \frac{d}{dz}M = \frac{F_s Q}{M}; \\ Q &= \int_0^\infty w_p d(r^2); \ M = \int_0^\infty w_p^2 d(r^2); \\ F_s &= \int_0^\infty g(1-\rho_p/\rho_a) w_p d(r^2), \end{aligned}$$

where ρ_p and w_p are the plume density and vertical velocity respectively, and α is the entrainment parameter which you may assume to be constant. Show that these equations have an attracting similarity solution, to which all plumes with source conditions $M(0) = M_s$ and $Q(0) = Q_s$ are attracted. If both $Q_s > 0$ and $M_s > 0$, derive a relationship between these source conditions such that the plume is in *pure plume balance* at all heights, a concept you should define carefully.

Consider two identical *top-hat* plumes, with source radii b_s, vertical velocities w_s and reduced gravities g'_s, such that the plumes are in pure plume balance at their respective sources. The centres of the two sources are separated by a distance L_s. Assuming that the two plumes do not interact as they rise, use an equation for the plume radius to determine the height z_m at which the edge of one plume is predicted to be vertically above the midpoint of the source of the other plume. Calculate the plume radius b_m, velocity w_m and reduced gravity g'_m at this height in terms of F_s, Q_s and L_s.

Assume that the two plumes combine at this height, so that the combined plume has velocity w_m, reduced gravity g'_m and radius $\sqrt{2}b_m$ (i.e. the cross-sectional area $2\pi b_m^2$ is the sum of the two independent plumes). Determine whether this combined plume is *forced*, *lazy* or *pure*, concepts you should define carefully. Identify the cross-sectional area πb_p^2 such that the combined plume is in pure plume balance, assuming that the reduced gravity and vertical velocity are not affected by the merging process, and comment briefly on your results.

3

Consider a two-dimensional inviscid flow with a piecewise linear mean velocity distribution $\overline{U}(z)$ and piecewise constant mean density distribution $\overline{\rho}(z)$. You may assume that the perturbation streamfunction ψ,

$$\psi(x,z,t) = Re\left[\hat{\psi}(z)\exp(ik[x-ct])\right]$$

(the wavenumber k is real) satisfies $\hat{\psi}'' = k^2\hat{\psi}$ except at 'interfaces' where the mean velocity, density or vorticity is discontinuous. At such interfaces, you may also assume that $\hat{\psi}$ satisfies the jump conditions:

$$\left[\frac{\hat{\psi}}{(\overline{U}-c)}\right]_-^+ = 0; \;\left[(\overline{U}-c)\frac{d}{dz}\hat{\psi} - \hat{\psi}\frac{d}{dz}\overline{U} - \frac{g\overline{\rho}}{\rho_0}\left(\frac{\hat{\psi}}{(\overline{U}-c)}\right)\right]_-^+ = 0.$$

Using a characteristic length scale h, velocity difference ΔU, density difference $\Delta\rho$ and reference density ρ_0 to scale quantities:

$$c = \frac{\Delta U}{2}\tilde{c};\; \overline{U} = \frac{\Delta U}{2}\tilde{U};\; \overline{\rho} = \rho_0 + \frac{\Delta\rho}{2}\tilde{\rho};\; z = \frac{h\tilde{z}}{2};\; \alpha = \frac{kh}{2};\; J = \frac{g\Delta\rho h}{\rho_0 \Delta U^2};$$

the nondimensional form of the jump conditions are

$$\left[\frac{\tilde{\psi}}{(\tilde{U}-\tilde{c})}\right]_-^+ = 0; \;\left[(\tilde{U}-\tilde{c})\frac{d}{d\tilde{z}}\tilde{\psi} - \tilde{\psi}\frac{d}{d\tilde{z}}\tilde{U} - J\tilde{\rho}\left(\frac{\tilde{\psi}}{(\tilde{U}-\tilde{c})}\right)\right]_-^+ = 0.$$

Consider a three-layer flow:

$$\tilde{U} = \begin{cases} 1 \\ \tilde{z} \\ -1 \end{cases}, \qquad \tilde{\rho} = \begin{cases} -1 & \tilde{z} > 1; \\ 0 & |\tilde{z}| < 1; \\ 1 & \tilde{z} < -1. \end{cases}$$

Show that $\tilde{c}$ satisfies

$$\tilde{c}^4 + \tilde{c}^2\left[\frac{e^{-4\alpha} - (2\alpha-1)^2}{4\alpha^2} - 1 - \frac{J}{\alpha}\right] + \left[\frac{(2\alpha - [1+J])^2 - e^{-4\alpha}(1+J)^2}{4\alpha^2}\right] = 0.$$

Hence show that the flow is unstable for

$$\frac{\alpha e^{\alpha}}{\cosh\alpha} < 1 + J < \frac{\alpha e^{\alpha}}{\sinh\alpha}.$$

Interpret this instability in terms of a wave resonance in the limit of large wavenumber.

4

Consider a cylindrical tank, of depth H and radius R. Defining the upper surface as $z = 0$, initially there is a linear density stratification ranging from density ρ_t at $z = 0$ to $\rho_b > \rho_t$ at $z = -H$. The upper surface of the tank is a disc which is driven at a steady rotation rate Ω. Assume that this disc drives mixing such that an upper layer of uniform density $\rho_u(t)$ and depth $h(t)$ develops, while the linearly stratified lower layer remains stationary. Show that the potential energy of the system increases by an amount

$$\Delta P = \rho_0 \pi R^2 N^2 \frac{h^3}{12}; \quad N^2 = \frac{g(\rho_b - \rho_t)}{\rho_0 H},$$

where N is the buoyancy frequency and ρ_0 is a reference density. Show that the reduced gravity jump at the interface between the well-mixed upper layer and the linearly stratified lower layer is given by

$$\frac{g(\rho_i - \rho_t)}{\rho_0} - \frac{g(\rho_u - \rho_t)}{\rho_0} = \frac{N^2 h}{2},$$

where ρ_i is the density at the interface at the top of the linearly stratified lower layer. For some drag coefficient c_D, the interfacial stress is $\tau = c_D \rho_0 u_u^2$, where u_u is a characteristic (radial) velocity in the upper layer. Briefly present physical arguments to justify the assumption that the kinetic energy of the upper well-mixed layer will approach a constant.

It is experimentally observed that the characteristic thickness δ of the interface between the well-mixed upper layer and the linearly stratified lower layer is constant. You should further assume that:

1: the rate of increase of potential energy is proportional to the rate of working of the turbulent stress at the interface, multiplied by a nondimensional power law function proportional to $(\Omega/N)^\alpha (\delta/R)^\beta$ (α and β to be determined);

2: the constant kinetic energy does not depend on N, and hence $u_u^2 h = A\Omega^2 R^3$ for some empirical constant A;

3: the entrainment parameter E depends only on the local interfacial Richardson number Ri_i, i.e.

$$E = \frac{dh/dt}{u_u} \propto (Ri_i)^{-\gamma}, \quad Ri_i = \frac{g(\rho_i - \rho_u)\delta}{\rho_0 u_u^2},$$

where γ is once again to be determined.
Hence establish that the layer depth h is expected to scale as

$$\frac{h}{R} = B \left(\frac{\Omega^2 R}{N^2 \delta} \right)^{1/3} (\Omega t)^{2/9},$$

for some empirical constant B.

END OF PAPER