UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 9:00 am to 12:00 pm

PAPER 7

TOPICS IN KINETIC THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In this question we will sometimes use the notation $f_t(x,v) = f(t,x,v)$. Consider the inhomogeneous transportation equation:

$$\begin{cases} \partial_t f(t,x,v) + v\partial_x f(t,x,v) + x\partial_v f(t,x,v) = h(t,x,v) & x, v \in \mathbb{R}, t > 0 \\ f|_{t=0} = f_0. \end{cases} \tag{1}$$

(a) Solve the characteristic equations associated with (1) and assuming $f_0 \in C^1(\mathbb{R} \times \mathbb{R})$ and $h \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ find an explicit solution to the equation (you are required to show your steps and not just give a final answer).

(b) Show explicitly that $\det\left(\frac{\partial(S_{0,t}(x,v))}{\partial(x,v)}\right) = 1$ for all $t > 0$ and conclude that if $h = 0$ then

$$\|f_t\|_{L^p(\mathbb{R}\times\mathbb{R})} = \|f_0\|_{L^p(\mathbb{R}\times\mathbb{R})}$$

for all $t > 0$ and $p > 0$.

(c) Show that in the case where $h \neq 0$ we have that for any $t > 0$

$$\|f_t\|_{L^\infty(\mathbb{R}\times\mathbb{R})} \leqslant \|f_0\|_{L^\infty(\mathbb{R}\times\mathbb{R})} + \|h\|_{L^\infty(\mathbb{R}\times\mathbb{R})}\, t,$$

and give an example where equality is attained.

(d) Give an example for $h \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, and $f_0 \in C^1(\mathbb{R} \times \mathbb{R}) \cap \left(\cap_{p\geqslant 1} L^p(\mathbb{R} \times \mathbb{R})\right)$ such that

$$\|f_t\|_{L^p(\mathbb{R}\times\mathbb{R})} = \infty,$$

for all $t > 0$ and $p \geqslant 1$.

(e) Assuming that $h \not\equiv 0$, give additional conditions on h such that the above can't happen (not necessary optimal).

2

In this question we will show that the phenomena of phase mixing discussed in class can be improved under additional regularity conditions. For simplicity we will only deal with the one dimensional case. We will sometimes use the notation $f_t(x,v) = f(t,x,v)$. Consider the transportation equation:

$$\begin{cases} \partial_t f(t,x,v) + v\partial_x f(t,x,v) = 0 & x \in \mathbb{T}, v \in \mathbb{R}, t \in \mathbb{R} \\ f|_{t=0} = f_0. \end{cases} \tag{1}$$

Assume that there exists $n \in \mathbb{N}$ such that $f_0 \in C^1(\mathbb{T}\times\mathbb{R})$ and f_0 is $1+n$ times differentiable in its velocity variable. Denote by

$$\|f_0\|_{L^1_x W^{1+n,1}_v} = \sum_{k=0}^{n+1} \int_{\mathbb{T}} \int_{\mathbb{R}} \left| \frac{\partial^k f_0}{\partial v^k} \right| (x,v)\,dx\,dv$$

and assume that $\|f_0\|_{L^1_x W^{1+n,1}_v} < \infty$. Define the following quantities:

$$\rho_t(x) = \int_{\mathbb{R}} f_t(x,v)\,dv,$$
$$\rho_\infty = \int_{\mathbb{T}} \rho_0(x)\,dx.$$

Our goal is to show that $\rho_t(x)$ converges to ρ_∞ uniformly.

(a) Show that $\rho_t \in L^1(\mathbb{R})$ for all t and that

$$\rho_\infty = \int_{\mathbb{T}} \rho_t(x)\,dx$$

for all $t \in \mathbb{R}$.

(b) Rewrite equation (1) for $\widehat{f}(t,k,\xi)$, $k \in \mathbb{Z}, \xi \in \mathbb{R}$, the Fourier transform of f in its spatial and velocity variables. Write an explicit solution to the new equation.

(c) Define $r_t(x) = \rho_t(x) - \rho_\infty \in L^1(\mathbb{T})$. Find the Fourier coefficient of r_t, $\widehat{r_t}(k)$, and express it only with $\widehat{f_0}$.

(d) Use the fact that $\left|\widehat{f}\right| \leqslant \|f\|_{L^1}$ to show that

$$\sup_k \left\{ \left| \widehat{f_0}(k,kt) \right| |kt|^{n+1} \right\} \leqslant \frac{\|f_0\|_{L^1_x W^{1+n,1}_v}}{(2\pi)^{n+1}}.$$

(e) Show that there exists C, depending only on n and f_0, such that

$$\sum_k |\widehat{r_t}(k)| \leqslant \frac{C}{|t|^{n+1}},$$

and conclude, using the fact that if $\widehat{f} \in L^1$ then $\|f\|_{L^\infty} \leqslant \left\|\widehat{f}\right\|_{L^1}$, that

$$\lim_{t\to\infty} \|\rho_t(x) - \rho_\infty\|_{L^\infty(\mathbb{T})} = 0,$$

where the order of convergence is $O\left(\frac{1}{|t|^{n+1}}\right)$.

3

This question is dedicated to the proof of existence and uniqueness of weak solution to a particular linear Boltzmann equation. We will sometimes use the notation $f_t(x,v) = f(t,x,v)$.
Consider the equation:

$$\begin{cases} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) = \rho(f)(t,x)M(v) - f(t,x,v) & x,v\in\mathbb{R}^d, t>0 \\ f|_{t=0}(x,v) = f_0(x,v), \end{cases} \tag{1}$$

where

$$\rho(f)(t,x) = \int_{\mathbb{R}^d} f_t(x,v)dv.$$

and $M(v)$ is a non negative function with $\int_{\mathbb{R}^d} M(v)dv = 1$. We say that $f(t,x,v)$ is an L^1 weak solution to (1) if $f_t \in L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ for any $t>0$ and

$$\begin{aligned} f(t,x,v) = f_0(x-vt,v) + \int_0^t \left(\rho(f)(s,x-v(t-s))M(v) - f(s,x-v(t-s),v)\right) ds \\ = F\left(f_0\right)(t,x,v) + \tau(f)(t,x,v). \end{aligned} \tag{2}$$

with $F\left(f_0\right)(t,x,v) = f_0(x-vt,v)$ and

$$\tau(f)(t,x,v) = \int_0^t \left[\rho(f)(s,x-v(t-s))M(v) - f(s,x-v(t-s),v)\right] ds.$$

(a) Assuming that $f_0 \in L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ show that

$$\left\| F\left(f_0\right)(t,\cdot,\cdot)\right\|_{L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)} = \left\|f_0\right\|_{L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)},$$

for all $t>0$.

(b) Show that

$$\int_{\mathbb{R}^d} \left|\rho(f)(t,x)\right| dx \leqslant \left\|f_t\right\|_{L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)},$$

and conclude that

$$\left\|\tau(f)(t,\cdot,\cdot)\right\|_{L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)} \leqslant 2\sup_{0\leqslant s\leqslant t}\left\|f_s\right\|_{L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)} t,$$

for all $t>0$.

(c) Show that for all $n\in\mathbb{N}$

$$\left\|\tau^n(f)(t,\cdot,\cdot)\right\|_{L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)} \leqslant \sup_{0\leqslant s\leqslant t}\left\|f_s\right\|_{L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)} \frac{2^n t^n}{n!},$$

for all $t>0$.

(d) Conclude the existence of an L^1 weak solution to (1) when $f_0 \in L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)$.

(e) Show that the L^1 weak solution you constructed satisfies:

$$\sup_{0 \leqslant s \leqslant t} \|f_s\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} < \infty \tag{3}$$

for all $t > 0$

(f) Under the additional condition (3) on L^1 weak solutions prove that the L^1 weak solution to (1) is unique.

4

The Boltzmann equation is one of the fundamental equation of Kinetic Theory. Unfortunately, to this day there is no proof of its validity in macroscopic time scales from Newtonian mechanics. In this question we will present a simplistic version of Kac's model, introduced in 1956. Kac's model is an attempt to create a stochastic many particle linear model from which, under certain conditions, the Boltzmann equation arises as a limit of the marginal as the number of the particles goes to infinity.
Consider a non-negative function $F_N \in L^2\left(\mathbb{R}^N\right)$, with $N \geqslant 1$, satisfying

$$\int_{\mathbb{R}^{\mathbb{N}}} F_N\left(v_1, \ldots, v_N\right) dv_1 \ldots dv_N = 1$$

(i.e. a probability density on $\mathbb{R}^N$ that also belongs to L^2). Consider the linear operator

$$QF_N\left(\mathbf{v}\right) = \frac{1}{\begin{pmatrix} N \\ 2 \end{pmatrix}} \sum_{i<j} \frac{1}{2\pi} \int_0^{2\pi} F_N\left(R_{i,j,\theta}(\mathbf{v})\right) d\theta,$$

where $\mathbf{v} = (v_1, \ldots, v_N)$ and

$$R_{i,j,\theta}(\mathbf{v}) = (v_1, \ldots, v_{i-1}, v_i(\theta), v_{i+1}, \ldots, v_{j-1}, v_j(\theta), v_{j+1}, \ldots, v_N)$$

with

$$v_i(\theta) = v_i \cos\theta + v_j \sin\theta, \quad v_j(\theta) = -v_i \sin\theta + v_i \cos\theta.$$

(a) Show that Q is a bounded self adjoint operator on $L^2\left(\mathbb{R}^N\right)$.
Hint: Use the rotation invariance of $\mathbb{R}^N$.

(b) Show that

$$\langle F_N, (I-Q)F_N \rangle_{L^2(\mathbb{R}^N)} = \frac{1}{4\pi} \frac{1}{\begin{pmatrix} N \\ 2 \end{pmatrix}} \sum_{i<j} \int_0^{2\pi} \left|F_N\left(R_{i,j,\theta}(\mathbf{v})\right) - F_N(\mathbf{v})\right|^2 d\theta$$

and conclude that $\mathrm{Ker}\,(I-Q)$ is exactly all the L^2 function that are radial (i.e. depend only on $|\mathbf{v}|$).

The $N-$particle model we study in this question satisfies the evolution equation

$$\partial_t F_N = N(Q-I)F_N \quad F_N \in L^2\left(\mathbb{R}^N\right) \tag{1}$$

which we will call Kac's master equation.
We will now see how the Boltzmann equation appears from Kac's model. From this point onward we will assume that F_N is symmetric in its variables, i.e.

$$F_N\left(v_1, \ldots, v_N\right) = F_N\left(v_{\sigma(1)}, \ldots, v_{\sigma N}\right),$$

for any permutation σ of $\{1, \ldots, N\}$. For $1 \leqslant k < N$, define the $k-$th marginal of F_N, $\Pi_k\left(F_N\right)$, as

$$\Pi_k\left(F_N\right)\left(v_1, \ldots, v_k\right) = \int_{\mathbb{R}^{N-k}} F_N\left(v_1, \ldots, v_N\right) dv_{k+1} \ldots dv_N.$$

(c) Show, assuming that all integration and differentiation is allowed, that the evolution equation for the first marginal, $\Pi_1(F_N)$ is given by

$$\partial_t \Pi_1(F_N)(v_1) = \frac{1}{\pi}\int_{\mathbb{R}}\int_0^{2\pi}\left(\Pi_2(F_N)(v_1(\theta), v_2(\theta)) - \Pi_2(F_N)(v_1, v_2)\right) dv_2 d\theta \qquad (2)$$

Hint: Use the rotation invariance of $\mathbb{R}^{N-1}$ *to cancel a lot of terms in the summation, and then use the symmetry of* F_N *to simplify the remaining terms.*

Equation (2) resembles the spatially homogeneous Boltzmann equation if $\Pi_2(F_N) \approx \Pi_1(F_N) \otimes \Pi_1(F_N)$. This is the basis to the definition of *chaos* in Kac's model, but we will leave it for the time being.

(d) Let $\gamma_N(\mathbf{v}) = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|\mathbf{v}|^2}{2}}$ be the Gaussian probability density in $\mathbb{R}^N$. Define

$$H_N(F_N) = \int_{\mathbb{R}^N} F_N(\mathbf{v}) \log\left(\frac{F_N(\mathbf{v})}{\gamma_N(\mathbf{v})}\right) d\mathbf{v}.$$

Using the fact that the function $f(x) = x\log x - x + 1$ is non negative for $x \geqslant 0$ show that $H_N(F_N) \geqslant 0$.

(e) Define the *entropy production* of F_N by

$$D_N(F_N) = -\frac{d}{dt} H_N(F_N),$$

where F_N satisfies (1). Assuming that all differentiation and integration is allowed, and that all quantities are defined, show that

$$D_N(F_N) = \frac{1}{2\pi(N-1)} \sum_{i<j} \int_{\mathbb{R}^N \times [0,2\pi]} \left(F_N(R_{i,j,\theta}(\mathbf{v})) - F_N(\mathbf{v})\right) \log\left(\frac{F_N(R_{i,j,\theta}(\mathbf{v}))}{F_N(\mathbf{v})}\right) d\mathbf{v} d\theta,$$

which has a definite sign.
Hint: Write

$$H_N(F_N) = \int_{\mathbb{R}^N} F_N(\mathbf{v}) \log(F_N(\mathbf{v})) d\mathbf{v} + \frac{N}{2}\log(2\pi) + \int_{\mathbb{R}^N} |\mathbf{v}|^2 F_N(\mathbf{v}) d\mathbf{v}.$$

You may use, without proving, that $\int_{\mathbb{R}^N} \phi(\mathbf{v})(I-Q) F_N(\mathbf{v}) d\mathbf{v} = 0$*, when* $\phi(\mathbf{v}) = 1$ *and* $\phi(\mathbf{v}) = |\mathbf{v}|^2$.

END OF PAPER