UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 5 June, 2014 9:00 am to 11:00 am

PAPER 69

THE UNIFIED METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS AND MEDICAL IMAGING

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $u(x,t)$ satisfy the PDE

$$\begin{aligned} &u_t = u_{xx} + \alpha u_x, && 0 < x < \infty, \quad 0 < t < T, \\ &u(x,0) = u_0(x), && 0 < x < \infty, \\ &u(0,t) = g_0(t), && 0 < t < T, \end{aligned}$$

where α and T are given positive constants, $u_0(x)$, $g_0(t)$ are given functions with sufficient smoothness, u_0 has sufficient decay as $x \to \infty$, and $u_0(0) = g_0(0)$.

(i) Rewrite the above PDE as a one parameter family of divergence forms and derive the associated global relation.

(ii) Derive an integral representation for $u(x,t)$ in the complex Fourier plane involving appropriate transforms of u_0, g_0 and $u_x(0,t)$ (the Jordan lemma can be used without proof).

(iii) Use the global relation to eliminate the transform of the unknown function $u_x(0,t)$, and hence obtain a solution of the above initial boundary value problem.

(iv) Prove that the expression for $u(x,t)$ obtained in (iii) satisfies the above PDE and the given initial and boundary conditions.

Can the above problem be solved via the classical sine transform?

2

Assuming the validity of the Poincare lemma

$$\int_{\partial D} F = \int_D dF, \tag{1}$$

where F is a one-form and D is an appropriate domain with regular boundary ∂D, implement the following:

(i) Derive the Pompeiu formula.

(ii) By employing the spectral analysis of the equation

$$\frac{\partial F(z,\overline{z},k,\overline{k})}{\partial \overline{z}} - kF(z,\overline{z},k,\overline{k}) = q(z,\overline{z}), \qquad z \in \mathbb{C}, k \in \mathbb{C}, \tag{2}$$

derive the two-dimensional Fourier transform pair.

(iii) Sketch, without giving all the mathematical details, the steps needed in order to derive the attenuated Radon transform, via the spectral analysis of

$$\frac{1}{2}\left(k + \frac{1}{k}\right)\frac{\partial F(x_1,x_2,k)}{\partial x_1} + \frac{1}{2i}\left(k - \frac{1}{k}\right)\frac{\partial F(x_1,x_2,k)}{\partial x_2} - \mu(x_1,x_2)F(x_1,x_2,k)$$
$$= f(x_1,x_2), \qquad (x_1,x_2) \in \mathbb{R}^2, \quad k \in \mathbb{C}.$$

3

(i) Show that $u(z,\overline{z})$ satisfies the modified Helmholtz equation

$$\frac{\partial^2 u(z,\overline{z})}{\partial z \partial \overline{z}} - \beta^2 u(z,\overline{z}) = 0, \qquad \beta > 0, z \in \mathbb{C}, \tag{1}$$

iff $dW = 0$, where

$$W(z,\overline{z},\lambda) = e^{-i\beta(\lambda z - \frac{\overline{z}}{\lambda})}\left[\left(\frac{\partial u}{\partial z} + i\beta\lambda u\right)dz - \left(\frac{\partial u}{\partial \overline{z}} + \frac{\beta u}{i\lambda}\right)d\overline{z}\right], \lambda \in \mathbb{C}\backslash\{0\}.$$

(ii) Let Ω be the interior of a polygon with corners $\{z_j\}_{j=1}^{N}$, indexed counter-clockwise modulo n. Let S_j denote the side parameterized by

$$z(s) = m_j + sh_j, \qquad -1 < s < 1, \quad j = 1, ..., n,$$

where m_j is the midpoint of the side. Show that

$$\sum_{j=1}^{n} \int_{-1}^{1} W_j(s,\lambda)ds = 0,$$

where an expression of W_j must be derived.

(iii) Let $z_1, ..., z_4$ be given by

$$z_1 = (1,i), \quad z_2 = (1,-i), \quad z_3 = (-1,-i), \quad z_4 = (-1,i).$$

Compute W_1, where S_1 denotes the side $(1,i),(1,-i)$.

Discuss in words without giving mathematical details, the steps needed for obtaining numerically the 4 unknown Neumann boundary values in terms of the given Dirichlet data.

END OF PAPER