UNIVERSITY OF CAMBRIDGE

## MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 1:30 pm to 3:30 pm

# PAPER 68

# IMAGE PROCESSING - VARIATIONAL AND PDE METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Let $1 < q < \infty$. Give a formal argument for the fact that functions $u \in W^{1,q}(0,1)$ are Hölder continuous. What does this imply for a function $u \in W^{1,q}((0,1)^2)$?

Define the space $BV((0,1)^2)$ and for a function $u \in BV((0,1)^2)$ the total variation $|Du|((0,1)^2)$. Give a formal derivation for the fact that for $u \in W^{1,1}((0,1)^2)$ we have

$$|Du|((0,1)^2) = \int_0^1 \int_0^1 |\nabla u| \; dx \; dy.$$

For $g \in L^2((0,1)^2)$ and $\alpha > 0$ consider the functional

$$\mathcal{J}(u) = \alpha \int_0^1 \int_0^1 \sqrt{1 + |Du|^2} \; + \; \frac{1}{2} \int_0^1 \int_0^1 (u - g)^2 \; dx \; dy,$$

where

$$\int_D \sqrt{1 + |Du|^2} = \sup \left\{ \int_D (\varphi_0 + u \mathrm{div} \varphi) \; dx \mid \varphi \in C_c^1(D; \mathbb{R}^2), \right.$$
$$\left. \varphi_0 \in C_c^1(D), \; |\varphi(x)| \leqslant 1, \; |\varphi_0(x)| \leqslant 1 \text{ for all } x \in D \right\}, \; D \subset \mathbb{R}^2 \text{ open},$$

and prove existence of a minimiser $u$ of $\mathcal{J}$ in $BV((0,1)^2)$.

For a minimiser $u \in W^{1,1}((0,1)^2)$ derive the corresponding Euler-Lagrange equation in the distributional sense over the space of compactly supported test functions.

**2**

Let $X = \mathbb{R}^{N \times N}$. For $g \in X$ and $\alpha > 0$ consider the minimisation problem

$$\min_{u \in X} \left\{ \alpha \|\nabla u\| + \frac{1}{2} \|u - g\|^2 \right\},$$

where $\nabla : X \to X^2$ is the discrete gradient operator and $\|\cdot\|$ is the usual Euclidean metric.

Find an expression for the minimiser $u$ as the projection of $g$ onto a closed, convex set. Carefully justify every step of your derivation and quote any definitions and theorems you use.

Give an iterative algorithm to compute this projection and state under which conditions this iteration converges.

Briefly describe the convex set associated with the projection that you get when you replace the gradient operator in the minimisation problem with the discrete Hessian operator $\nabla^2 : X \to X^4$.

**3**

Let $\Omega = (a,b) \times (c,d)$ be a rectangular image domain. For $g \in L^\infty(\Omega)$ and parameters $\alpha, \beta > 0$ consider the Mumford–Shah segmentation problem

$$\min_{(u,K)} \int_{\Omega \setminus K} (u-g)^2 \, dx + \alpha \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \beta \mathcal{H}^1(K).$$

Give an admissible set of minimisers for the above problem and outline an existence proof.

State the reduced problem for $\alpha \to +\infty$. Discuss what you get when minimising for $u$ and $K$ separately (fixing the other) in the reduced problem.

Finally, formally derive the reduced problem for $\beta \to +\infty$ and prove existence and uniqueness of minimisers.

**4**

Write an essay on image smoothing and edge enhancement with linear and nonlinear diffusion equations.

## END OF PAPER