UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 9:00 am to 11:00 am

PAPER 67

DISTRIBUTION THEORY AND APPLICATIONS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the convolution $*$ between $\mathcal{D}(\mathbf{R}^n)$ and $\mathcal{D}'(\mathbf{R}^n)$. Show that if $\varphi, \psi \in \mathcal{D}(\mathbf{R}^n)$ and $u \in \mathcal{D}'(\mathbf{R}^n)$ then

$$(u * \varphi) * \psi = u * (\varphi * \psi).$$

If $u, v \in \mathcal{D}'(\mathbf{R}^n)$, one of which has compact support, show that

$$(u * v) * \varphi := u * (v * \varphi), \quad \varphi \in \mathcal{D}(\mathbf{R}^n)$$

defines a unique element of $\mathcal{D}'(\mathbf{R}^n)$ and that $u * v = v * u$.

Suppose now that u and v both have compact support. Define the space of tempered distributions and the distributional Fourier transform. Explain why both u and v define tempered distributions and prove that

$$(u * v)\hat{}(\lambda) = \hat{u}(\lambda)\hat{v}(\lambda).$$

[*You may assume that if u has compact support then* $\hat{u}(\lambda) = \left\langle u(x), e^{-\mathrm{i}\lambda\cdot x}\right\rangle$.]

Consider the distribution $u_a \in \mathcal{D}'(\mathbf{R}^3)$ defined by the surface element on the sphere of radius $a > 0$, so that

$$\langle u_a, \varphi \rangle = a^2 \int_{S^2} \varphi(ax)\, \mathrm{d}\sigma_2, \quad \varphi \in \mathcal{D}(\mathbf{R}^3)$$

where $\mathrm{d}\sigma_2 \equiv \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$ is the surface element on the unit sphere $S^2 \subset \mathbf{R}^3$ in spherical polars $(x_1, x_2, x_3) \equiv (r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta)$. Show that

$$\hat{u}_a(\lambda) = \frac{4\pi a \sin(|\lambda| a)}{|\lambda|}.$$

Hence, show that

$$(u_a * u_b)(x) = \begin{cases} 2\pi a b |x|^{-1}, & |a-b| \leqslant |x| \leqslant a+b, \\ 0, & \text{otherwise.} \end{cases}$$

[*You may find it useful to use the following: for* $a, b, c > 0$

$$\frac{8}{\pi} \lim_{R\to\infty} \int_0^R \frac{\sin(ar)\sin(br)\sin(cr)}{r}\, \mathrm{d}r$$
$$= \operatorname{sgn}(a+b-c) + \operatorname{sgn}(b-a+c) - \operatorname{sgn}(b-a-c) - \operatorname{sgn}(a+b+c).]$$

2

Show that if $u \in \mathcal{E}'(\mathbf{R}^n)$ then $\hat{u}(\lambda) = \left\langle u(x), e^{-i\lambda\cdot x}\right\rangle$ and that this function can be extended to an entire function of $z \in \mathbf{C}^n$. If $\mathrm{supp}(u)$ is contained inside the closed ball $\{x \in \mathbf{R}^n : |x| \leqslant \delta\}$ establish the estimate

$$|\hat{u}(z)| \lesssim (1+|z|)^N e^{\delta|\mathsf{Im}\, z|}$$

for all $z \in \mathbf{C}^n$ and some $N \geqslant 0$.

Now let $U(z)$ be a complex valued, entire function of $z \in \mathbf{C}^n$. Suppose that U obeys an estimate of the form

$$|e^{iz\cdot y}U(z)| \lesssim (1+|z|)^N e^{\delta|\mathsf{Im}\, z|}$$

for some $N \geqslant 0$ and some fixed $y \in \mathbf{R}^n$. Prove that U is the Fourier transform of a distribution supported in $\{x \in \mathbf{R}^n : |x-y| \leqslant \delta\}$.

Consider the initial value problem

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \Delta_x E = 0 \qquad (x,t) \in \mathbf{R}^n \times (0,\infty)$$

$$E = f, \quad \frac{\partial E}{\partial t} = 0 \qquad \text{when } t = 0$$

where $f \in \mathcal{E}'(\mathbf{R}^n)$ has support contained in $\{x \in \mathbf{R}^n : |x-y| \leqslant \delta\}$. By treating t as a parameter, i.e. $E(x,t) \equiv E_t(x)$, show that for each $t > 0$ the support of $E(\cdot,t)$ is contained in the set

$$\{x \in \mathbf{R}^n : |x-y| \leqslant \delta + |c|t\}.$$

3

Let P_N be an Nth order polynomial in $\lambda = (\lambda_1, \ldots, \lambda_n)$. What does it mean to say that $P_N(D)$ is elliptic? Show that if P_N is elliptic then $|P_N(\lambda)| \gtrsim \langle\lambda\rangle^N$ for $|\lambda|$ sufficiently large.

Let $X \subset \mathbf{R}^n$ be open. Define the Sobolev space $H^s(\mathbf{R}^n)$ and local Sobolev space $H^s_{\mathrm{loc}}(X)$. Prove that if $u \in \mathcal{D}'(\mathbf{R}^n)$ has compact support then u belongs to a Sobolev space of sufficiently negative index.

Consider the linear differential operator with *non-constant* coefficients

$$L = P_N(D) + \sum_{|\alpha|<N} f_\alpha(x) D^\alpha$$

where each $f_\alpha \in C^\infty(X)$. Show that if P_N is elliptic and $Lu \in H^s_{\mathrm{loc}}(X)$ then $u \in H^{s+N}_{\mathrm{loc}}(X)$. Deduce that all solutions to $Lu = 0$ in X are smooth.

END OF PAPER