UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 29 May, 2014 9:00 am to 12:00 pm

PAPER 66

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Attempt no more than ***THREE*** *questions from Section A and* ***ONE*** *from Section B.*

There are ***SEVEN*** *questions in total.*

The questions in Section B carry twice the weight of those in Section A. Questions within each Section carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Let κ be a constant. We consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \kappa \frac{\partial u}{\partial x}, \qquad x \in [0,1], \quad t \geqslant 0,$$

given with initial conditions at $t = 0$ and zero Dirichlet boundary conditions at $x = 0$ and $x = 1$.

(a) Prove that the equation is well posed.

(b) The equation is semi-discretised by the ODE system

$$u_m' = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + \frac{\kappa}{2\Delta x}(u_{m+1} - u_{m-1}), \qquad m = 1, \ldots, M,$$

where $\Delta x = 1/(M+1)$ and $u_m(t) \approx u(m\Delta x, t)$. Carefully justifying your analysis, prove that the method is stable.

(c) The equation is fully discretised by the Crank–Nicolson method

$$\begin{aligned} u_m^{n+1} =& u_m^n + \frac{\mu}{2}(u_{m-1}^n - 2u_m^n + u_{m+1}^n) + \frac{\kappa\mu\Delta x}{4}(u_{m+1}^n - u_{m-1}^n) \\ &+ \frac{\mu}{2}(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) + \frac{\kappa\mu\Delta x}{4}(u_{m+1}^{n+1} - u_{m-1}^{n+1}), \qquad m = 1, \ldots, M, \quad n \geqslant 0, \end{aligned}$$

where $u_m^n \approx u(m\Delta x, n\Delta t)$ and $\mu = \Delta t/(\Delta x)^2$ is the Courant number. Prove that the method is stable for all $\mu > 0$.

[*Hint: You may use without a proof the fact that (in Euclidean norm)* $\|e^{tA}\| \leqslant e^{t\alpha[A]}$, $t \geqslant 0$, *where the scalar* $\alpha[A]$, *called the* spectral abscissa, *is the largest eigenvalue of the symmetric matrix* $\frac{1}{2}(A + A^\top)$, *and that this inequality is inherited by A-stable approximations to the exponential, that is* $\|r(tA)\| \leqslant r(t\alpha[A])$.]

2

Consider the two-step method

$$\mathbf{y}_{n+2} - (1+a)\mathbf{y}_{n+1} + a\mathbf{y}_n = \frac{1}{2}(1-3a)h\mathbf{f}(\mathbf{y}_{n+1}) + \frac{1}{2}(1+a)h\mathbf{f}(\mathbf{y}_{n+2}),$$

where a is a real parameter, for the solution of the ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$.

(a) Determine the order of this method for different values of a.

(b) For which values of a is the method convergent?

(c) For which values of a is the method A-stable?

3

(a) Prove that a collocation method with the distinct collocation points $c_1, \dots, c_s \in [0,1]$ is equivalent to the Runge–Kutta method with the nodes $c_1, \dots, c_s$, RK matrix

$$a_{m,k} = \int_0^{c_m} \ell_k(t)dt, \qquad m,k = 1,\dots,s$$

and the weights

$$b_k = \int_0^1 \ell_k(t)dt, \qquad k = 1,\dots,s,$$

where

$$\ell_k(t) = \prod_{\substack{m=1 \\ m\neq k}}^{s} \frac{t-c_m}{c_k-c_m}, \qquad k=1,\dots,s.$$

(b) Let $s=3$, $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = 1$. Find the Runge–Kutta method corresponding to these collocation points and determine its order.

(c) Is this method A-stable?

4

(a) Let A be an $d \times d$ symmetric matrix. Prove that (using Euclidean norm)

$$\|e^{tA}\| \leqslant e^{t\mu[A]}, \qquad t \geqslant 0,$$

where the *spectral abcissa* $\mu[A]$ is the largest eigenvalue of A, and that $\mu[A]$ is the smallest real number for which the above inequality is true.

(b) Consider the linear ODE system $\mathbf{y}' = (A+B)\mathbf{y}$, where the $d \times d$ matrices A and B are symmetric. We approximate its solution in the form $\mathbf{y}_{n+1} = F(t)\mathbf{y}_n$, $n = 0, 1, \ldots$, where

$$F(t) = \frac{1}{2}(e^{tA}e^{tB} + e^{tB}e^{tA}).$$

By considering the quantity $F'(t) - (A+B)F(t)$ and using variation of constants, or otherwise, prove that

$$F(t) - e^{t(A+B)} = \frac{1}{2}\int_0^t e^{(t-x)(A+B)}\{[e^{xB}, A]e^{xA} + [e^{xA}, B]e^{xB}\}dx,$$

where $[\,\cdot\,,\,\cdot\,]$ is the standard matrix commutator.

(c) Prove that

$$\|F(t) - e^{t(A+B)}\| \leqslant (\|A\| + \|B\|)\frac{e^{t(\mu[A]+\mu[B])} - e^{t\mu[A+B]}}{\mu[A] + \mu[B] - \mu[A+B]}, \qquad t \geqslant 0,$$

assuming that $\mu[A] + \mu[B] \neq \mu[A+B]$,

$$\|F(t) - e^{t(A+B)}\| \leqslant (\|A\| + \|B\|)te^{t\mu[A+B]}, \qquad t \geqslant 0,$$

otherwise.

5

The advection equation $\partial u/\partial t = \partial u/\partial x$ is solved by the two-step fully discretised method

$$u_m^{n+1} = (2\mu - 1)(u_{m+1}^n - u_m^n) + u_{m+1}^{n-1}, \qquad n \geqslant 1,$$

where $\mu = \Delta t/\Delta x$.

(a) Determine the order of the method.

(b) Determine the range of Courant numbers μ for which the method is stable for the Cauchy problem.

SECTION B

6

Write an essay on A-stability of numerical methods for initial-value ODEs. You should provide the overall background and definitions, comment on A-stability in different settings (rational, multistep and Runge–Kutta methods) and accompany your presentation with examples.

7

Write an essay on Ritz and Galerkin methods. You should define these methods within the broader framework of finite element methods, present their theoretical setting and justification formulating relevant conditions and theorems and comment about their implementation in the case of the two-point boundary value problem $-(pu')' + qu = f$, where $p > 0$, $q \geqslant 0$ and f are functions of appropriate smoothness.

END OF PAPER