UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 1:30 pm to 3:30 pm

# PAPER 64

# MEASURE AND IMAGE

*Attempt no more than* ***TWO*** *questions.*

*There are* ***THREE*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***

*Cover sheet*

*Treasury Tag*

*Script paper*

***SPECIAL REQUIREMENTS***

*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Answer the following questions concerning *weak* convergence in* BV($\Omega$), *and the direct method in the calculus of variations.*

(i) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. Characterise weak* convergence in BV($\Omega$), and state the criterion for a sequence in BV($\Omega$) to admit a weak* convergent subsequence.

(ii) Provide a proof of the above result (existence of a weak* convergent subsequence). The following Lemma from the lectures can be helpful:

**Lemma.** *Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a family of mollifiers, and $w \in \mathrm{BV}(\mathbb{R}^n)$ with compact support. Then*

$$\int_{\mathbb{R}^n} |(w * \rho_\epsilon)(x) - w(x)|\, dx \leqslant \epsilon |Dw|(\Omega).$$

(iii) Let $f \in L^1(\Omega)$ on a bounded open domain $\Omega$ with Lipschitz boundary. Let $K : L^1(\Omega) \to L^1(\Omega)$ be a bounded linear operator such that $K^{-1}$ is also bounded. Pick $\alpha > 0$. Show that the problem

$$\inf_{u \in \mathrm{BV}(\Omega)} \|f - Ku\|_{L^1(\Omega)} + \alpha |Du|(\Omega),$$

admits a solution, i.e, there exists a function $\widehat{u} \in \mathrm{BV}(\Omega)$ achieving the infimum value.

**2**

Answer the following questions concerning *the co-area formula in* BV(Ω)*, and its applications to the ROF denoising problem.*

(i) State the co-area formula for functions of bounded variation.

(ii) Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary, $\alpha > 0$, and $f \in L^2(\Omega) \cap \mathrm{BV}(\Omega)$. State and prove the level set formulation of the total variation regularisation problem

$$\min_{u \in \mathrm{BV}(\Omega)} \frac{1}{2}\|f-u\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}(u). \tag{1}$$

The following Lemma from the lectures can be helpful:

**Lemma.** *Suppose $h, g \in L^1(\Omega)$ with $g(x) < h(x)$ for $\mathcal{L}^n$-almost every $x \in \Omega$. If $\hat{E}$ and $\hat{F}$ solve, respectively*

$$\min_{E \subset \Omega} \mathrm{Per}(E;\Omega) - \int_E g(x)\,dx, \quad \textit{and} \quad \min_{F \subset \Omega} \mathrm{Per}(F;\Omega) - \int_F h(x)\,dx,$$

*then $\mathcal{L}^n(\hat{E} \setminus \hat{F}) = 0$.*

(iii) Let $\hat{u} \in \mathrm{BV}(\Omega)$ solve (1). State the main steps for proving that $\mathcal{H}^{n-1}(J_{\hat{u}} \setminus J_f) = 0$.

**3** Answer the following questions about *the structure and regularity properties of functions of bounded variation.*

(i) State precisely and prove the Structure Theorem in BV(Ω).

(ii) How can $Du$ be decomposed? Define and describe the different components.

(iii) Let $\Omega = (a,b) \subset \mathbb{R}$, and $u \in \mathrm{BV}(\Omega)$. Prove that there exist left- and right-continuous representatives of $u$, i.e., functions $u^l$ and $u^r$ such that $u^l(x) = u^r(x) = u(x)$ for $\mathcal{L}^1$-almost every $x \in (a,b)$, and $u^l$ is left- and $u^r$ right-continuous. Moreover, show that $u^l$ and $u^r$ are discontinuous on at most a countable collection of points.

*Hint:* Define $u^l(t) := c + Du((a,t))$ for suitable $c$. Prove that $u^l = u$ almost everywhere in $(a,b)$. In order to prove left-continuity, observe that $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ implies that $S_u \subset J_u$ for $n = 1$.

**END OF PAPER**