UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 1:30 pm to 3:30 pm

PAPER 63

ADVANCED QUANTUM INFORMATION THEORY

*Attempt no more than **TWO** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A *stoquastic circuit* is a quantum circuit consisting of 2- and 3-qubit gates which are all permutation matrices in the computational basis.

The input to the stoquastic circuit is always a computational basis state, together with a set A_0 of ancilla qubits each in the $|0\rangle$ state, and a set A_+ of ancilla qubits each in the $|+\rangle$ state, where we define the single qubit states $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

The output of the stoquastic circuit is the result of measuring the first qubit in the $\{|+\rangle, |-\rangle\}$ basis.

The complexity class StoqMA is the class of decision problems for which there exists a polynomial-sized *stoquastic* verifier circuit U acting on a computational basis state $|w\rangle$ such that:

$$\begin{cases} \exists\,|w\rangle : \Pr(U \text{ outputs "+" on input } |w\rangle) \geqslant \frac{2}{3} & \text{YES instance} \\ \forall\,|w\rangle : \Pr(U \text{ outputs "+" on input } |w\rangle) \leqslant \frac{1}{3} & \text{NO instance} \end{cases}$$

A *stoquastic Hamiltonian* is a Hamiltonian whose off-diagonal matrix elements in the computational basis are all $\leqslant 0$.

(a) Use the Kitaev construction with non-local clock to write down a Hamiltonian H encoding a stoquastic verifier circuit (*without* any penalty term on the output of the circuit).

Show that H is stoquastic.

(b) Prove that the 0-energy eigenstates of H are computational history states for the stoquastic verifier circuit.

You may use without proof the fact that the matrix $E = \frac{1}{2}\sum_{t=0}^{T}(|t\rangle - |t+1\rangle)(\langle t| - \langle t+1|)$ has eigenvalues $\lambda_k = 1 - \cos q_k$ where $q_k = \frac{\pi k}{T+1}$, $k = 0, \ldots, T$, and that the eigenvector corresponding to λ_0 is $\frac{1}{\sqrt{T+1}}\sum_{t=0}^{T}|t\rangle$.

(c) Prove that the spectral gap $\Delta(H) = \Omega(T^{-3})$.

You may use without proof the following variant of Kitaev's geometrical lemma (where $\operatorname{supp} X$ denotes the support of an operator X):
Let $A \geqslant 0$, $B \geqslant 0$ be positive-semidefinite operators satisfying $\lambda_{\min}(A|_{\operatorname{supp} A}) \geqslant \mu$ and $\lambda_{\min}(B|_{\operatorname{supp} B}) \geqslant \mu$. The smallest non-zero eigenvalue of $A + B$ satisfies

$$\lambda_{\min}\Big((A+B)|_{\operatorname{supp}(A+B)}\Big) \geqslant 2\mu \sin^2\frac{\theta}{2}$$

where $\theta := \max_{\substack{|\xi\rangle \in \ker A \\ |\eta\rangle \in |B\rangle \\ |\eta\rangle \perp K}} \langle \eta|\xi\rangle$ with $K = \ker A \cap \ker B$.

You may also use without proof all the properties of the matrix E given in part (b), and the relation $\sin^2\frac{\theta}{2} \geqslant \frac{1-\cos^2\theta}{8\cos^2\theta}$ valid for $0 \leqslant \theta \leqslant \frac{\pi}{4}$.

(d) Let $H' = H + \delta H_{\text{out}}$, where $H_{\text{out}} := |-\rangle\langle -|_1 \otimes |T\rangle\langle T|$ and $\delta > 0$ is a constant.

By considering H', or otherwise, prove that the local Hamiltonian problem for stoquastic Hamiltonians (with non-local clock) is StoqMA-hard.

You may use without proof the following perturbation bound for arbitrary Hamiltonians H and V, valid for any $\delta < \Delta(H)$:

$$\lambda_{\min}(H + \delta V) = \delta \min_{\psi \in \ker H} \langle \psi | H_{\text{out}} | \psi \rangle + O(\delta^2).$$

[*Hint: Show first that for a computational history state* $|\psi\rangle$ *for circuit input* $|w\rangle$, $\langle\psi|H_{out}|\psi\rangle = \frac{1}{T+1}\Big(1 - \Pr(U \textit{ outputs "+" on input } |w\rangle)\Big)$.]

2

(a) State the definition of the complexity class BQP, and explain why the class does not depend on the precise values chosen for the probabilities appearing in the definition.

(b) Define the complexity class C to be the class of decision problems for which there exists a classical algorithm that, for problem size n, solves the problem using an arbitrary number of basic arithmetic operations on real numbers (addition, subtraction, multiplication), but only ever stores at most poly(n) real numbers in memory.

Show that BQP $\subseteq$ C.

[*Hint: The probability of measuring* $\Pi_1^{(1)}$ *on the output of circuit* U *running on input* $|x\rangle$ *can be computed as*

$$\sum_{i_2, i_3, \ldots, i_n = 0,1} \Big| \langle 1| \langle i_2| \cdots \langle i_n| \ U \ |x\rangle \Big|,$$

and the product $U_1 U_2$ *can be rewritten as*

$$U_1 U_2 = \sum_{i_1, i_2, \ldots, i_n = 0,1} U_1 \ |i_1\rangle |i_2\rangle \cdots |i_n\rangle \langle i_1| \langle i_2| \cdots \langle i_n| \ U_2. \quad]$$

(c) State the definition of the complexity class QMA, and show that QMA $\subseteq$ C.

You may use without proof the following inequality for the maximum eigenvalue $\lambda_{\max}(X)$ of an n-qubit operator X:

$$\ln \lambda_{\max}(X) \leqslant \frac{1}{d} \ln \mathrm{Tr}(X^d) \leqslant \ln \lambda_{\max}(X) + \frac{n}{d} \ln 2.$$

[*Hint: Consider computing the maximum probability (over all witnesses) that the verifier circuit outputs "1".*]

3 For a system of qudits, let A_X and B_Y be operators that act non-trivially only on qudits in the sets X and Y. $A_X(t)$ denotes the time-evolution of A_X in the Heisenberg picture.

Let Z range over k-element subsets of the qudits, and let $H = \sum_Z h_Z$ be a k-local Hamiltonian where h_Z acts non-trivially only on the subset Z.

(a) Suppose that for each qudit i in the system, $\sum_{Z:Z\ni i} \|h_Z\| \leqslant se^{-\mu}$ for some constants $\mu, s > 0$ (where $\|X\|$ denotes the operator norm of X).

Starting from the integral equation

$$C_A(Y,t) \leqslant C_A(Y,0) + 2 \sum_{Z:Y\cap Z\neq\emptyset} \|h_Z\| \int_o^t C(Z,s)\,\mathrm{d}s$$

where (for operators O_Y acting non-trivially only on Y)

$$C_A(Y,t) := \sup_{O_Y} \frac{\|[A_X(t), O_Z]\|}{\|O_Y\|},$$

prove the Lieb-Robinson bound:

$$\|[A_X(t), B_Y]\| \leqslant 2\,\|A_X\|\;\|B_Y\|\;\min(|X|,|Y|)\;e^{-\mu d(X,Y)}\left(e^{2kst} - 1\right),$$

where $d(X,Y)$ denotes interaction distance.

(b) In this part of the question, you may use without proof the following corollary of the Lieb–Robinson bound:

$$\|A_X(t) - A_{X'}(t)\| \leqslant \mu vt|X|\,\|A_X\|e^{-\mu l/2},$$

where $A_{X'}(t)$ acts non-trivially only on the set of qudits $X' = \{i : d(i,X) \leqslant vt + l\}$, and $v = 4ks/\mu$ is the Lieb-Robinson velocity.

Let $L = \mathrm{diam}(S)$ be the total diameter of the many-body system on which H acts (where diameter $\mathrm{diam}(X) := \max_{i,j\in X} d(i,j)$.)

A state $|\psi_1\rangle$ has *topological quantum order* if there exists another state $|\psi_2\rangle$ with the following properties:

i. $\langle\psi_2|\psi_1\rangle = 0$

ii. For any A_X with $\|A_X\| \leqslant 1$ and $\mathrm{diam}(X) \leqslant cL$:
$|\langle\psi_1|A_X|\psi_1\rangle - \langle\psi_2|A_X|\psi_2\rangle| \leqslant \epsilon$

for some constants $c, \epsilon < 1$.

Let $|\psi_0\rangle$ be a state which evolves under H into $|\psi_0(t)\rangle = |\psi_1\rangle$.

Prove that if $t = cL$ for a sufficiently small constant $c > 0$, then $|\psi_0\rangle$ itself must be topologically ordered. (You are *not* required to find an explicit expression for c.)

[*Hint: Consider a second state* $|\tilde{\psi}_0\rangle$ *that evolves into* $|\psi_2\rangle$*.*]

4 In this question, you may use without proof the fact that there exists a function $w(t)$ with Fourier transforms $\hat{w}(E)$ satisfying:

$$w(t) \geqslant 0, \qquad \hat{w}(E) = 0 \text{ for } E \geqslant \Delta, \qquad \hat{w}(0) = 1, \quad \text{and}$$
$$\int_T^\infty w(t)\mathrm{d}t = O\left((\ln \beta T)^2 \, e^{-\beta T/(\ln \beta T)^2}\right) \text{ for some constants } c, \beta > 0.$$

For a system of qudits, let Z range over k-element subsets of the qudits and let $H = \sum_Z h_Z$ be a k-local Hamiltonian, where h_Z acts non-trivially only on the subset Z.

You may assume that H satisfies the Lieb-Robinson bound

$$\|[A_X(t), B_Y]\| \leqslant 2 \, \|A_X\| \, \|B_Y\| \, \min(|X|, |Y|) \, e^{-\mu d(X,Y)} \left(e^{2kst} - 1\right)$$

for operators A_X and B_Y that act non-trivially only on qudits in the subsets X and Y. Here, $A_X(t)$ denotes the time-evolution of A_X under H in the Heisenberg picture, and $d(X,Y)$ denotes the interaction distance between subsets X and Y.

(a) Assume that H has a unique ground state $|\phi_0\rangle$, eigenstates $|\phi_i\rangle$, and spectral gap $\Delta > 0$.

Explain how the Fourier transform and filtering technique can be used to construct an operator $A^{(Z)}$ with matrix elements:

i. $\langle\phi_0|A^{(Z)}|\phi_0\rangle = \langle\phi_0|h_Z|\phi_0\rangle$,

ii. $\langle\phi_0|A^{(Z)}|\phi_i\rangle = \langle\phi_i|A^{(Z)}|\phi_0\rangle = 0$ for $i > 0$.

(b) Using the Fourier transform and filtering technique, or otherwise, prove that H can be rewritten as $H = \sum_Z A^{(Z)}$ with $[A^{(Z)}, P_0] = 0$, where $P_0 := |\phi_0\rangle\langle\phi_0|$.

(The operators $A^{(Z)}$ may act on all the qudits in the system.)

(c) Let

$$A^{(Z,d)} := \int_{-\infty}^{\infty} w(t) \left(e^{itH_d} \, h_Z \, e^{-itH_d} - e^{itH_{d-1}} \, h_Z \, e^{-itH_{d-1}}\right) \mathrm{d}t,$$

where $H_d := \sum_{Y\,:\,d(Z,Y)\leqslant d} h_Y$, so that $H_0 = h_Z$ and $H_D = H$.

By summing $A^{(Z,d)}$ over d, or otherwise, show that $A^{(Z)} = h_Z + \sum_{d=1}^{D} A^{(Z,d)}$ for suitably chosen $A^{(Z)}$ from part (b).

(d) Assume that $\sum_{Y:d(Z,Y)=d} \|h_Y\| \min(|Y|,|Z|) = O(d^\alpha)$ for some constant $\alpha > 0$. Starting from the relation

$$\left\|e^{itH_d} h_Z e^{-itH_d} - e^{itH_{d-1}} h_Z e^{-itH_{d-1}}\right\| \quad \leqslant \quad \sum_{Y:d(Z,Y)=d} \int_0^t \|[h_Z(\tau), h_Y]\| \, \mathrm{d}\tau$$

where $h_Z(\tau) := e^{i\tau H_d}\, h_Z\, e^{-i\tau H_d}$, prove that

$$\left\|e^{itH_d}\, h_Z\, e^{-itH_d} - e^{itH_{d-1}}\, h_Z\, e^{-itH_{d-1}}\right\| \quad \leqslant \quad O\left(\|h_Z\| t\, d^\alpha\, e^{-\mu d + 2kst}\right).$$

(e) Using part (d), or otherwise, prove that $\left\|A^{(Z,d)}\right\| \leqslant O\left(\|h_Z\| d^{\alpha+1}\, e^{-cd/\ln^2 cd}\right)$ for some constant $c > 0$. (You are *not* required to find an explicit expression for c.)

END OF PAPER