UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 5 June, 2014 9:00 am to 11:00 am

PAPER 62

QUANTUM FOUNDATIONS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State the Schmidt decomposition theorem. (You do not need to prove it.)

Hence, or otherwise, show that any pure state of two separated qubits can be written in the form

$$|\psi\rangle = \lambda_0 |0\rangle_A |0\rangle_B + \lambda_1 |1\rangle_A |1\rangle_B \,,$$

where $\{|0\rangle_A, |1\rangle_A\}$, $\{|0\rangle_B, |1\rangle_B\}$ are orthonormal bases for the two factor spaces and the λ_i are real and positive.

Explain why, with appropriate separate choices of coordinates for the separate subsystems, we may take the states $\{|0\rangle, |1\rangle\}$ to be eigenstates of σ_z for each subsystem.

Show that the expected value of a measurement of the operator $\mathbf{a} \cdot \boldsymbol{\sigma} \otimes \mathbf{b} \cdot \boldsymbol{\sigma}$ in the state $|\psi\rangle$, where $\mathbf{a}, \mathbf{b}$ are any two vectors in $\mathbb{R}^3$, is

$$P(\mathbf{a}, \mathbf{b}) = \langle \mathbf{a} \cdot \boldsymbol{\sigma} \otimes \mathbf{b} \cdot \boldsymbol{\sigma} \rangle_\psi = 2\lambda_0 \lambda_1 (a_x b_x - a_y b_y) + a_z b_z \,.$$

Hence show that for suitable choices of measurement axes $\mathbf{a}, \mathbf{a}'$ for the A subsystem and $\mathbf{b}, \mathbf{b}'$ for the B subsystem, the state $|\psi\rangle$ violates the CHSH inequality. Comment briefly on the significance of this result.

2

In all parts of this question you may restrict your discussion to the case of ideal preparation and measurement devices (i.e. devices that operate with perfect precision and are error free).

Explain briefly the statement of the Pusey-Barrett-Rudolph (PBR) theorem. Also explain briefly in words what it assumes about state preparation, and the hypothesis about quantum states that it refutes (given the stated assumptions).

Prove the PBR theorem for the special case of two qubit states, $|0\rangle$ and $|+\rangle$, whose inner product $\langle 0 \,|\, + \rangle = 1/\sqrt{2}$.

Now consider two qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$ whose inner product $\langle \psi_1 \,|\, \psi_2 \rangle = (1/\sqrt{2})^{1/n}$. By considering independent preparations of $2n$ copies of the states $|\psi_i\rangle$, or otherwise, prove the PBR theorem for the special case of states $|\psi_1\rangle$ and $|\psi_2\rangle$.

3

A given quantum system has Hamiltonian H and is isolated, apart from the action of experimental measuring devices. Starting from the standard time-asymmetric formulation of quantum theory, derive time-symmetric expressions for the outcome probabilities of a projective measurement defined by a complete set of projectors P_j at time t in a sub-ensemble of experiments on the system defined by pre-selecting the state $|\psi\rangle$ at time t_1 and post-selecting the state $|\psi'\rangle$ at time t_2, where $t_1 < t < t_2$.

Suppose now that the system is $(N+1)$-dimensional, and its Hilbert space has an orthonormal basis $\{|i\rangle : i = 1, 2, \ldots, N+1\}$. Suppose also that the Hamiltonian $H = 0$ and that the pre- and post-selected states are

$$|\psi\rangle = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} |i\rangle \,,$$

$$|\psi'\rangle = \frac{1}{\sqrt{N^2 - N + 1}} \Big(\sum_{i=1}^{N} |i\rangle - (N-1)|N+1\rangle\Big) \,.$$

For each i in the range $1 \leqslant i \leqslant N$, consider the experiment E_i in which the complete set of projectors is given by $P_i = |i\rangle\langle i|$ and $1 - P_i$. What are the corresponding outcome probabilities in experiment E_i for the given pre- and post-selected subensemble?

Now consider an experiment E_0 in which the complete set of projectors is given by $P_1, P_2, \ldots, P_{N+1}$. What are the corresponding outcome probabilities in experiment E_0 for the given pre- and post-selected sub-ensemble?

Comment briefly on any apparently counter-intuitive features of your answers.

4

Describe the equations of motion and the rules for spontaneous localisation "collapses" for a discrete Ghirardi–Rimini–Weber (GRW) model for N distinguishable particles with no internal degrees of freedom, in terms of the average single particle collapse time τ and the collapse scale a.

Suppose we model a superposition of macroscopically distinct pointer states by supposing the pointer is made out of N distinguishable particles, initially in a state

$$\begin{aligned}\psi(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}) &= \alpha\phi(\mathbf{x_1} - \mathbf{y_1})\phi(\mathbf{x_2} - \mathbf{y_2})\ldots\phi(\mathbf{x_N} - \mathbf{y_N}) + \\ &\quad \beta\phi(\mathbf{x_1} - \mathbf{z_1})\phi(\mathbf{x_2} - \mathbf{z_2})\ldots\phi(\mathbf{x_N} - \mathbf{z_N})\,.\end{aligned}$$

Here $\phi(\mathbf{x})$ is a normalised position space wave function with the property that $\phi(\mathbf{x})$ is negligible for $|\mathbf{x}| > r$, where the distance $r > a$, and $|\alpha|^2 + |\beta|^2 = 1$. The points $\mathbf{y_i}$ lie within a region Y, and the points $\mathbf{z_i}$ lie within a region Z, and the regions Y and Z are macroscopically separated, so that $|\mathbf{y} - \mathbf{z}| \gg 100r$ (and hence also $|\mathbf{y} - \mathbf{z}| \gg 100a$) for any $\mathbf{y} \in Y$ and $\mathbf{z} \in Z$.

In the following calculations you may neglect the Schrödinger evolution during timescales relevant for this problem, i.e. you may take the Hamiltonian to be zero. You may also ignore small terms provided you clearly explain why they are small.

What is the probability that the first spontaneous collapse of one of the particles takes place in or near (say, no further than $10a$ from) region Y? Given that the first spontaneous collapse does take place in or near region Y, what is the probability that the next spontaneous collapse of one of the particles also takes place in or near region Y? Explain how your answers follow from the GRW model. You should give the form of any relevant integrals and explain why they are approximated by your answers, but need not compute the integrals precisely.

Comment briefly on the physical significance of these calculations.

END OF PAPER