UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 1:30 pm to 4:30 pm

PAPER 60

QUANTUM INFORMATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Define the fidelity $F(\rho,\sigma)$ of two states ρ and σ. Give an expression for the fidelity of two *pure* states φ and ψ.

(ii) Clearly state Uhlmann's theorem. Use it to prove monotonicity of the fidelity under partial trace.

(iii) Let ρ denote the density matrix corresponding to an ensemble of states $\{p_x,\rho_x\}_{x=1}^m$. For each $x \in \{1,2,\ldots,m\}$, let $|\psi_{\rho_x}\rangle$ denote a purification of ρ_x. Find a purification $|\psi_\rho\rangle$ of ρ in terms of the $|\psi_{\rho_x}\rangle$.

(iv) Using the above, prove joint concavity of the fidelity, i.e.,

$$F\left(\sum_{x=1}^m p_x\rho_x, \sum_{x=1}^m p_x\sigma_x\right) \geqslant \sum_{x=1}^m p_x F\left(\rho_x,\sigma_x\right).$$

2

(i) (a) Consider a memoryless classical channel with input and output alphabets given by $J = \{1, 2, 3\}$, and transition matrix

$$p(y|x) = \begin{pmatrix} q_1 & q_2 & q_3 \\ q_3 & q_1 & q_2 \\ q_2 & q_3 & q_1 \end{pmatrix},$$

where $q_i > 0$ for each $i \in \{1, 2, 3\}$, and $q := \{q_1, q_2, q_3\}$ forms a probability distribution. If X and Y are the random variables corresponding respectively to the input and output of this channel, find an expression for $H(Y|X)$ in terms of the entropy of the probability distribution q.

[*Hint: Notice that the transition matrix is bistochastic.*]

(b) Find an expression for the capacity of the channel, clearly stating the relevant theorem.

(ii) Let $X \sim p(x)$ be a random variable taking values in a finite, discrete alphabet J_X with probability mass function $p(x)$. Suppose we make an inference about X based on knowledge of a random variable Y. Suppose our best guess of X is given by a random variable $f(Y)$ (which is a function of Y and takes values in J_X). The probability p_e of an erroneous inference is given by

$$p_e = P\left(f(Y) \neq X\right).$$

(a) If Z is the indicator function $Z = I(X \neq f(Y))$, express the Shannon entropy of Z in terms of the entropy of the probability distribution $\{p_e, 1 - p_e\}$.

(b) Verify that the conditional entropy $H(Z, X|Y) := H(X, Y, Z) - H(Y)$ can be expanded as follows:

$$H(Z, X|Y) = H(X|Y) + H(Z|X, Y). \tag{1}$$

Similarly find an alternative expansion of $H(Z, X|Y)$ as a sum of two conditional entropies – one of them being $H(X|Z, Y)$.

(c) Using these two expansions of $H(Z, X|Y)$ prove that

$$H(X|Y) \leqslant p_e \log\left(|J_X| - 1\right) + h(p_e)$$

(where $h(p_e) := -p_e \log p_e - (1 - p_e)\log(1 - p_e)$ denotes the binary entropy), justifying each step clearly.

[*Hint: Note that $H(Z|X, Y) = 0$ since Z is a function of X and $f(Y)$, and use the expansion*

$$H(X|Z, Y) = P(Z = 0)H(X|Y, Z = 0) + P(Z = 1)H(X|Y, Z = 1).$$]

3

(i) Define the Choi–Jamiołkowski state of a completely positive trace-preserving (CPTP) map Λ which maps states in a Hilbert space $\mathcal{H}$ to states in a Hilbert space $\mathcal{K}$.

(ii) A completely dephasing map Λ, with respect to a complete orthonormal basis $\mathcal{B} := \{|y\rangle\}$ in a Hilbert space $\mathcal{H}$, acts on a state ρ on the Hilbert space $\mathcal{H}$ as follows:

$$\Lambda(\rho) = \sum_y |y\rangle\langle y|\rho|y\rangle\langle y|.$$

Prove that Λ is a CPTP map.

(iii) Find an expression for the von Neumann entropy $S(\sigma)$ of the state $\sigma := \Lambda(\rho)$.

(iv) Prove that $S(\sigma) \geqslant S(\rho)$ [*Hint: Use Klein's inequality*].

4

(i) State the necessary and sufficient condition for a bipartite pure state $|\psi_{AB}\rangle$ to be transformed to another bipartite pure state $|\phi_{AB}\rangle$ by local operations and classical communication (LOCC) alone. Clearly explain any particular notation that you use.

(ii) Using the above, prove that the Schmidt rank of a bipartite pure state cannot be increased by LOCC alone.

(iii) State the strong subadditivity property of the von Neumann entropy. Expressing the property equivalently as an inequality between two relative entropies, justify it using the monotonicity of the relative entropy under partial trace.

(iv) Let AB denote a composite quantum system which is in a state ρ_{AB}. Let Λ denote a completely positive trace-preserving (CPTP) map acting on the subsystem B alone, and let $\sigma_{A'B'} = (\mathrm{id} \otimes \Lambda)(\rho_{AB})$ denote the state of the composite system after this action. Here id denotes the identity map. Using the strong subadditivity of the von Neumann entropy, prove that

$$I(A' : B')_\sigma \leqslant I(A : B)_\rho,$$

where $I(A : B)_\rho$ denotes the quantum mutual information of the state ρ_{AB}.

5

Let ρ be the state of a quantum system A and let $|\psi^{\rho}_{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$ denote a purification of ρ. Let Λ_1 denote a quantum channel which acts on states of the system A to yield states of a system B_1. We denote this as $\Lambda_1 : A \to B_1$. The coherent information of the channel Λ_1, with respect to the input state ρ is defined as:

$$I_c\left(\Lambda_1, \rho\right) := S(\sigma_{B_1}) - S(\sigma_{RB_1}),$$

where $\sigma_{RB_1} := (\mathrm{id}_R \otimes \Lambda_1)\, \psi^{\rho}_{RA}$, with $\psi^{\rho}_{RA} = |\psi^{\rho}_{RA}\rangle\langle\psi^{\rho}_{RA}|$, and $S(\cdot)$ denoting the von Neumann entropy.

(i) Prove that the coherent information can equivalently be expressed as a conditional entropy between the systems R and E, where E denotes the environment in the Stinespring dilation of the channel Λ_1. State your reasons clearly.

(ii) Prove that $I_c\left(\Lambda_1, \rho\right) \leqslant S(\rho)$.

(iii) If $\Lambda_2 : B_1 \to B_2$ denotes another quantum channel, prove that

$$I_c\left(\Lambda_2 \circ \Lambda_1, \rho\right) \leqslant I_c\left(\Lambda_1, \rho\right).$$

(iv) Using the above, prove the following quantum data-processing inequality for the quantum mutual information:

$$I(R : B_1)_\sigma \geqslant I(R : B_2)_\omega,$$

where $\omega_{RB_2} := (\mathrm{id}_R \otimes \Lambda_2 \circ \Lambda_1)\, \psi^{\rho}_{RA}$.

END OF PAPER