UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 4 June, 2014 9:00 am to 12:00 pm

PAPER 6

FUNCTIONAL ANALYSIS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Throughout this question work with real scalars.

What is meant by a *locally convex space*? Define the topology of a locally convex space. [You do not need to verify that this is a topology.]

Define the dual space X^* of a locally convex space X and prove that it separates the points of X. [No version of the Hahn–Banach theorem can be assumed without proof.]

Let Y be a closed subspace of a locally convex space X, and let $x_0 \in X \setminus Y$. Show that there exists $f \in X^*$ with $f(x_0) = 1$ and $f(y) = 0$ for all $y \in Y$.

2

Throughout this question X is a real Banach space and you are free to use any version of the Hahn–Banach theorem (both extension and separation theorems) without proof.

(i) Define the w^*-topology on the dual space X^*. State Goldstine's Theorem.

(ii) Let E be a finite-dimensional subspace of X^*, let $x^{**} \in X^{**}$ with $\|x^{**}\| \leqslant 1$, and let $\varepsilon > 0$. Prove that there is an $x \in X$ such that $\|x\| < 1 + \varepsilon$ and $x^{**}(e) = e(x)$ for all $e \in E$. Briefly explain how to use this result to prove Goldstine's Theorem.

(iii) Let Y be a closed, finite-codimensional subspace of X^*. Briefly explain how to find finitely many elements $\phi_1, \phi_2, \ldots, \phi_n$ of X^{**} such that $Y = \bigcap_{k=1}^{n} \ker\phi_k$. Set $F = \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$ and assume that Y is w^*-dense in X^*. Show that $F \cap X = \{0\}$. (Here we identify X with its canonical image in X^{**}.) Show that for some $c > 0$, Y is *c-norming for X*:

$$c\|x\| \leqslant \sup\{x^*(x) : x^* \in Y, \ \|x^*\| \leqslant 1\} \qquad \text{for all } x \in X \ .$$

[*Hint: First show that the distance d between S_X (the unit sphere of X) and F is positive:*

$$d = d(S_X, F) = \inf\{\|x - \phi\| : x \in S_X, \ \phi \in F\} > 0 \ .$$

Given $x \in S_X$, apply (ii) to $E = span(F \cup \{x\})$ and a suitable functional on X^{**} to show that $c = d$ works.]

(iv) Show that a subspace Z of X^* that is c-norming for X for some $c > 0$ is necessarily w^*-dense in X^*. Give an example of a Banach space X and a closed subspace Z of X^* that is 1-norming for X and has infinite codimension in X^*.

3

Throughout this question K is a compact Hausdorff space, and $C(K)$ is the Banach space of continuous complex-valued functions on K with the uniform norm.

State the Riesz Representation Theorem. Explain *very briefly* how to deduce a characterization of the dual space of $C(K)$.

Let X be a complex Banach space, and let C be a non-empty, w^*-compact, convex subset of the dual space X^*. Define what is meant by an *extreme point of* C. State the Krein–Milman Theorem for C. Assume that $S \subset C$ and the w^*-closure of the convex hull of S is C. Prove that the w^*-closure of S contains all extreme points of C. [The Hahn-Banach separation theorems may be used without proof.]

Identify with proof the set of extreme points of the closed unit ball of $C(K)^*$. [You may use without proof any fact about faces of w^*-compact, convex sets.]

Let F be a finite-dimensional subspace of $C[0,1]$. Given a complex, Borel measure μ on $[0,1]$ with $\|\mu\|_1 = 1$, and given $\varepsilon > 0$, show that there exist $N \in \mathbb{N}$, $w_1, w_2, \dots, w_N \in [0,1]$, and complex numbers $t_1, t_2, \dots, t_N$ with $\sum_{i=1}^N |t_i| = 1$ such that

$$\Big| \int_0^1 f \,\mathrm{d}\mu - \sum_{i=1}^N t_i f(w_i) \Big| \leqslant \varepsilon \|f\| \qquad \text{for all } f \in F\ .$$

4

Let A be a complex, unital, commutative Banach algebra. Let $x \in A$ and U be an open subset of $\mathbb{C}$ that contains the spectrum $\sigma(x)$ of x. Show that there is a unique, continuous, unital algebra homomorphism $\Theta_x \colon \mathcal{O}(U) \to A$ such that $\Theta_x(u) = x$, where $u \in \mathcal{O}(U)$ is the function $u(z) = z,\ z \in U$. Show further that for each $f \in \mathcal{O}(U)$ we have $\sigma\big(\Theta_x(f)\big) = \{f(\lambda) : \lambda \in \sigma(x)\}$. [You may assume standard results from complex analysis and vector-valued integration as well as elementary properties of invertible elements and elementary spectral theory in Banach algebras. You may also assume Runge's approximation theorem provided it is clearly stated.]

Let X be a Banach space and $T \colon X \to X$ be a bounded linear map. Assume that the spectrum of T is disconnected. Show that there is a proper, non-trivial closed subspace Y of X that is invariant under T, *i.e.*, $Ty \in Y$ for all $y \in Y$. [*Hint: Find a projection* $P \colon X \to X$ *that is not zero or identity with* $PT = TP$.]

5

Let A be a complex commutative, unital Banach algebra.

What is a *character* on A? Show that characters on A are continuous. [You may assume results about invertible elements.]

Define the *spectrum* Φ_A of A. Define the *Gelfand topology* of Φ_A, and prove that Φ_A is compact in the Gelfand topology. Show also that the spectrum of $x \in A$ is $\sigma(x) = \{\phi(x) : \phi \in \Phi_A\}$. [You may assume results about the w^*-topology and the Gelfand–Mazur Theorem about normed division algebras.]

Now let A be a commutative, unital C^*-algebra. Show that the spectral radius of $x \in A$ is $r(x) = \|x\|$. Show also that $\phi(x^*) = \overline{\phi(x)}$ for any $x \in A$ and $\phi \in \Phi_A$. [You may assume the spectral radius formula and other results from elementary spectral theory.]

State and prove the Gelfand–Naimark Theorem.

END OF PAPER