UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 9:00 am to 12:00 pm

PAPER 59

PLANETARY SYSTEM DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A planet of mass M and density ρ orbits a star of mass $M_\star$ at a constant distance a. Find the semimajor axis $a_{\rm s}$ of planet-synchronous orbits, which are circumplanetary orbits with an orbital period equal to the planet's rotation period P, and comment on why such orbits are favoured for Earth's communications satellites.

Show that such orbits are possible providing that

$$\sqrt{\frac{3\pi}{\rho G}} < P < \sqrt{\frac{4\pi^2 a^3}{3GM_\star}},$$

where G is the gravitational constant.

The inhabitants of the planet start launching spherical satellites of radius r onto planet-synchronous orbits with randomly oriented orbital planes and small eccentricities e. Show that the mean time between collisions amongst a population of $N_{\rm s}$ such satellites is $AN_{\rm s}^{-2}$, where

$$A = P\frac{e}{4}\left(\frac{a_{\rm s}}{r}\right)^2.$$

Comment on how this collision timescale would be affected if the satellites had been launched onto near planet-stationary orbits, prograde orbits with small mean inclinations I from the planet's equator, where $1 \gg I \gg e$?

If the satellite population grows linearly at a rate R, show that the first satellite collision is likely to occur when the number of satellites reaches $N_{\rm s0} = (3AR)^{1/3}$.

Two of the satellites collide at the time determined above at which point no more satellites are launched. The collision destroys both objects with their mass redistributed into spherical fragments of radius xr, where $x \ll 1$. Assume the fragments' orbits to have the same isotropic distribution to that of the rest of the satellite population. Determine the conditions on x and $N_{\rm s0}$ for the next collision amongst the populations of satellites and fragments to be expected to occur between two fragments.

Fragment-fragment collisions can be assumed to remove the mass of both fragments from the system, while fragment-satellite collisions remove the fragment's mass but break the satellite into fragments of radius xr. Show that

$$\dot{N}_{\rm f} = A^{-1}[2x^{-3}N_{\rm s}^2 + x^{-3}N_{\rm f}N_{\rm s}/4 - x^2N_{\rm f}^2],$$

where $N_{\rm f}$ and $N_{\rm s}$ are the number of fragments and satellites respectively, and find the corresponding expression for $\dot{N}_{\rm s}$.

Assuming that satellite-satellite collisions can be ignored, discuss how the fragment population evolves, quantifying where possible.

2

Consider a source on a circular orbit at a distance of a from a star of mass $M_\star$. A particle created by the source is given a velocity of $\gamma v_{\rm k}$ (on top of the Keplerian velocity of the source $v_{\rm k}$) in a direction that is at an angle θ to the motion of the source in its orbital plane. Show that the semimajor axis a' of the particle's orbit is given by

$$a'/a = (1 - 2\gamma\cos\theta - \gamma^2)^{-1}.$$

Find the corresponding expression for the particle's eccentricity e', and show that its minimum possible value is

$$\min(e') = \gamma\sqrt{1 - \frac{\gamma^2}{3} + \frac{\gamma^4}{27}},$$

which occurs at $\cos\theta = -\gamma/3$.

Particles are created with θ randomly distributed between 0 and 2π. Sketch the distribution of particle orbits on a plot of e' versus a'/a, quantifying the locations of any extrema in this distribution.

If ϖ' is the longitude of the particle's orbit relative to the point at which it was created, show that

$$e'\cos\varpi' = (2 + \gamma\cos\theta)\gamma\cos\theta.$$

Particles placed on orbits that are close to mean motion commensurabilities (i.e., where the orbital periods are a ratio of two integers) are susceptible to close encounters with the source. If a close enounter is defined as the particle coming within a distance Δ of the source, show that the fraction of particles placed on orbits that have close encounters with the source due to the 1:1 mean motion commensurability is given by

$$\frac{\Delta}{6\pi^2 a\gamma\sqrt{1-\gamma^2/4}}.$$

Sketch the orbits of particles in the frame rotating with the source for $\gamma \ll 1$ and for $\theta = 0, \pi, \pi/2$.

3

A test particle orbits a star of mass $M_\star$ with a semimajor axis a, eccentricity e and longitude of pericentre ϖ. The particle's orbital plane is the same as that of a planet of mass $M_{\rm pl} \ll M_\star$ which is on a circular orbit interior to that of the particle at a distance $a_{\rm pl}$ from the star. The particle is close to the $p+q:p$ resonance for which the resonant argument is $\phi = (p+q)\lambda - p\lambda_{\rm pl} - q\varpi$, where p and q are integers, $q \leqslant 2$, and λ and $\lambda_{\rm pl}$ are the mean longitude of the particle and planet respectively. Identifying terms in the disturbing function involving ϕ as well as secular terms up to second order in e, Lagrange's planetary equations give for the evolution of the particle's orbital elements

$$\begin{aligned} \dot{a} &= -2(p+q)e^q a C_{\rm r} \sin\phi, \\ \dot{e} &= -qe^{q-1} C_{\rm r} \sin\phi, \\ \dot{\varpi} &= 2C_{\rm s} + qe^{q-2} C_{\rm r} \cos\phi, \end{aligned}$$

where $C_{\rm r}$ and $C_{\rm s}$ are both positive constants that are $\propto M_{\rm pl}/M_\star$. What conditions must hold for these equations to provide a good description of the particle's motion?

Ignoring changes in mean longitude at epoch, derive the full equation of motion for $\ddot{\phi}$ and show that there are two fixed points at $\phi = 0, \pi$.

Give physical reasons for which of these points is stable both for $q = 1$ and $q = 2$.

The particle starts at $\phi = \pi + \Delta\phi$ with $\dot{\phi} = 0$, where $\Delta\phi \ll 1$. Show that the particle's semimajor axis is at approximately

$$a_{\rm r}\left[1 - \frac{2}{3pn_{\rm pl}}\left(2qC_{\rm s} - q^2 C_{\rm r} e^{q-2}\right)\right],$$

where $a_{\rm r}$ is the nominal location of the resonance and $n_{\rm pl}$ is the mean motion of the planet.

Ignoring terms that are second order or higher in small quantities, derive expressions for the time evolution of ϕ, a, e and ϖ.

Sketch the motion on plots of $\dot{\phi}$ versus ϕ and e versus a, quantifying where possible.

Consider now the case where $\Delta\phi$ is no longer small. Show that

$$E = \frac{1}{2}\dot{\phi}^2 + 2\omega_0^2 \sin^2\left(\frac{\phi - \pi}{2}\right)$$

is a constant, where ω_0 should be determined.

With the aid of a plot of $\dot{\phi}$ versus ϕ describe how the particle's motion changes as $\Delta\phi$ approaches π.

4

Secular interactions cause the complex eccentricities of a coplanar two planet system to evolve according to

$$\dot{\boldsymbol{z}} = iA\boldsymbol{z}, \qquad (*)$$

where $\boldsymbol{z} = [z_1, z_2]$, $z_j = e_j \exp i\varpi_j$ is the complex eccentricity of the j-th planet, e_j and ϖ_j are the eccentricity and longitude of pericentre of the j-th planet, and A is a matrix with positive diagonal elements A_{11} and A_{22}, and negative off-diagonal elements A_{12} and A_{21}. Show that the two eigenvalues of A are

$$\begin{aligned} \lambda_p &= \frac{1}{2}[A_{11} + A_{22} + \sqrt{(A_{11} - A_{22})^2 + 4A_{12}A_{21}}], \\ \lambda_m &= \frac{1}{2}[A_{11} + A_{22} - \sqrt{(A_{11} - A_{22})^2 + 4A_{12}A_{21}}]. \end{aligned}$$

Show that the solution to equation $(*)$ can be written

$$z_j = e_{jp} \exp i(\phi_{jp} + \lambda_p t) + e_{jm} \exp i(\phi_{jm} + \lambda_m t),$$

where $e_{jp} \exp i(\phi_{jp})$ and $e_{jm} \exp i(\phi_{jm})$ are constants set by the initial conditions, and derive the following relations

$$\begin{aligned} e_{1p}/e_{2p} &= (A_{22} - \lambda_p)/A_{21}, \\ \phi_{1p} &= \phi_{2p} + \pi, \\ e_{1m}/e_{2m} &= (\lambda_m - A_{22})/A_{21}, \\ \phi_{1m} &= \phi_{2m}. \end{aligned}$$

The planets start at $\varpi_1 = 60^\circ$ and $\varpi_2 = 150^\circ$, with $\phi_{1p} = 30^\circ$ and $\phi_{1m} = 120^\circ$. Sketch their location in the complex eccentricity plane, showing the combination of the modes associated with the two eigenvalues, and describe how their complex eccentricities evolve.

Given that $A_{12}/A_{21} \approx L_1/L_2$, where L_j is the angular momentum in the j-th planet, show that

$$(e_{1p}/e_{2p})(e_{1m}/e_{2m}) = L_1/L_2.$$

If tides act to reduce the eccentricity of planet 1 according to

$$\dot{e}_1 = -e_1/\tau,$$

where τ is a constant timescale that is large compared with secular timescales (i.e., $\tau \gg 1/\lambda_m$), show that the evolution of the complex eccentricities of this system can still be written in the form of equation $(*)$ but with a revised matrix A'.

Find the eigenvalues of A' and describe how tides affect the evolution of the complex eccentricities of the two planets.

END OF PAPER