UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 4 June, 2014 1:30 pm to 4:30 pm

PAPER 58

GALACTIC ASTRONOMY AND DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

An axisymmetric disc galaxy has the property that a star in circular orbit of radius R has velocity

$$v(R) = v_0,$$

where R is cylindrical polar radius, and v_0 is a constant. Show that the surface density of the disc, assumed infinitesimally thin, is

$$\Sigma(R) = \frac{v_0^2}{2\pi G}\frac{1}{R}\,\delta(z),$$

and that the three-dimensional potential is

$$\phi(r, z) = v_0^2 \log(r + |z|),$$

where r is spherical polar radius and z is height above or below the disc.

Explain what is meant by *the distribution function* F of stars in the disc. Demonstrate that the isotropic distribution function F depends on the energy E of stellar orbits only.

Show that $F(E)$ satisfies the integral equation

$$\Sigma(R) = 2\pi \int_{\phi}^{\infty} F(E)dE.$$

By regarding Σ as a function of ϕ, solve the integral equation and show that

$$F(E) = \frac{1}{4\pi^2 G}\exp(-E/v_0^2).$$

Find the velocity dispersions of stars in the disc.

Show that the mean rotational velocity of the stars vanishes. If all the counter-rotating stars have their azimuthal velocities reversed, what is the new streaming velocity and why is it not equal to v_0?

[*Hint: You are reminded of the standard integral* ($\alpha > 0$)

$$\int_{-\infty}^{\infty} \exp(-\alpha v^2)dv = \sqrt{\frac{\pi}{\alpha}} \quad]$$

2

A piece of a galaxy rotating with uniform angular velocity Ω is modelled as a sheet of gas in which the planar pressure P varies as Σ^γ, where Σ is the surface density and γ is a constant. Stating carefully any assumptions, show that the dispersion relation for waves of wavenumber k and angular frequency ω is

$$\omega^2 = k^2c^2 - 2\pi G\Sigma_0|k| + 4\Omega^2,$$

where $c^2 = \gamma P_0/\Sigma_0$.

For a non-rotating sheet, show that perturbations with wavenumber $|k| < 2\pi G\Sigma_0/c^2$ are unstable.

For a rotating sheet with zero pressure, show that perturbations with wavenumber $|k| > 2\Omega^2/(\pi G\Sigma_0)$ are unstable.

If both rotation and pressure operate, derive the condition for instability as

$$\frac{\Omega c}{G\Sigma_0} \geqslant \frac{\pi}{2}.$$

At what wavelength does instability arise if c^2 is slowly decreased, and what is the characteristic size of the pieces into which the sheet breaks up?

3

A Newtonian self-gravitating system comprises N point masses m_k located at positions $\underline{r}_k$ relative to the centre of mass and moving with velocities $\underline{v}_k = \dot{\underline{r}}_k$. The kinetic energy T, potential energy U and moment of inertia $2I$ of the system are defined as

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k |\underline{v}_k|^2,$$
$$I = \frac{1}{2}\sum_{k=1}^{N} m_k |\underline{r}_k|^2,$$
$$U = \sum_{1\leqslant j<k\leqslant N} \frac{Gm_j m_k}{r_{jk}},$$

where $r_{jk} = |\underline{r}_j - \underline{r}_k|$. Prove the virial theorem in the form

$$\frac{d^2 I}{dt^2} = 2T - U = T + E,$$

where $E = T - U$ is the total energy.

Show that an alternative expression for I is

$$I = \frac{1}{2M} \sum_{1\leqslant j<k\leqslant N} m_j m_k r_{jk}^2,$$

where M is the total mass.

Let r denote the minimum spacing between particles (i.e., $r_{jk} \geqslant r$ for all pairs j,k) and R denote the maximum spacing (i.e., $r_{jk} \leqslant R$ for all pairs j,k) at some instant in time. Show that $U \leqslant A_0 r^{-1}$ for some A_0 dependent on the masses, and that $U \geqslant Gm_1m_2r^{-1}$ where m_1 and m_2 are the two smallest masses. Hence, deduce that U^{-1} is a measure of r in the sense that there exist positive constants A_0 and B_0 depending on the masses such that

$$B_0 \leqslant rU \leqslant A_0.$$

Show further that I is a measure of R^2 in the sense that there exist positive constants A_1 and B_1 depending on the masses such that

$$B_1 R^2 \leqslant I \leqslant A_1 R^2.$$

Demonstrate that if $E < 0$, then the minimum spacing between particles is bounded below.

Finally, show that if $E > 0$, the maximum spacing R increases at least as fast as the first power in time.

4

A particle with position vector $\underline{r}$ moves in a Keplerian potential $\phi(r) = -GM/r$. Show that the energy E and angular momentum $\underline{L}$ are integrals of the motion, where

$$\begin{aligned} \underline{L} &= \underline{r} \times \underline{\dot{r}}, \\ \frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} &= E - \phi(r). \end{aligned} \qquad (1)$$

Here, dots denote time derivatives, while $L = |\underline{L}|$.

Show additionally that the components of the Laplace-Runge-Lenz vector

$$\underline{A} = \underline{\dot{r}} \times \underline{L} - \frac{GM\underline{r}}{r}$$

are also conserved.

By transforming to new variables $\bar{r} = \bar{r}(r)$ and $\bar{t} = \bar{t}(t)$, and denoting differentiation with respect to $\bar{t}$ by primes, show that eq.(1) takes the form

$$\frac{1}{2}\bar{r}'^2 + \frac{L^2}{2\bar{r}^2} = \frac{r^2}{\bar{r}^2}\left(E + \frac{GM}{r}\right), \qquad (2)$$

provided

$$\frac{r}{\bar{r}} = \left(\frac{d\bar{r}}{dr}\right)\left(\frac{dt}{d\bar{t}}\right).$$

By identifying the last term on the right-hand side of eq. (2) as the new energy $\bar{E}$, show that the new potential is

$$\bar{\phi}(\bar{r}) = -E\left(\frac{\bar{E}}{GM}\right)^2 \bar{r}^2.$$

This therefore provides a mapping from orbits in the Kepler potential to the harmonic oscillator.

If instead the transformed energy is taken as λ times the old potential and $1 - \lambda$ times the old energy, show that

$$\begin{aligned} \bar{E} &= \left(\frac{r}{\bar{r}}\right)^2 \left((1-\lambda)\frac{GM}{r} + \lambda E\right), \\ \bar{\phi}(\bar{r}) &= -\left(\frac{r}{\bar{r}}\right)^2 \left((1-\lambda)E + \lambda\frac{GM}{r}\right). \end{aligned}$$

Now, find $r = r(\bar{r})$ and show that this defines a mapping from orbits in the Kepler potential to the isochrone

$$\bar{\phi}(\bar{r}) = -\frac{G\bar{M}}{b + \sqrt{\bar{r}^2 + b^2}} + k, \qquad (3)$$

where k is a constant and

$$b = \frac{(1-\lambda)GM}{2\sqrt{\lambda E\bar{E}}}.$$

[*Remark: You are not required to find explicit expressions for $\bar{M}$ or k in eq. (3).*]

END OF PAPER