

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 1:30 pm to 3:30 pm

PAPER 57

DYNAMICS OF ASTROPHYSICAL DISCS

Attempt no more than **TWO** questions.

There are **THREE** questions in total.

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

**You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.**

1 Photoevaporation of a protoplanetary disk

The ideal gas in the surface layer of a thin disk becomes irradiated by its central star so that it attains a temperature of 10^4 K, which corresponds to a sound speed $c \approx 10$ km s $^{-1}$. You may assume that such gas is in circular Keplerian orbits while this occurs.

(a) At the gravitational radius r_g the sound speed of the gas in the surface layer equals its orbital speed. Show that $r_g = GM/c^2$, and explain why the gas beyond this radius is gravitationally unbound and may leave the disk in a wind. (Here M is the mass of the star.)

If $M_\odot \sim 2 \times 10^{33}$ g, $G \sim 7 \times 10^{-8}$ cm 3 g $^{-1}$ s $^{-2}$, and 1 AU $\sim 10^{13}$ cm, give an estimate for r_g in the protosolar nebula (in AU).

(b) Mass and angular momentum conservation in the photoevaporated disk are governed by the equations

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r} \frac{\partial \mathcal{F}_{\text{acc}}}{\partial r} - \mathcal{W}, \quad \mathcal{F}_{\text{acc}} = 6\pi r^{1/2} \frac{\partial}{\partial r} (r^{1/2} \nu \Sigma)$$

where \mathcal{F}_{acc} is the radial mass accretion rate and \mathcal{W} is the mass loss rate due to the photoevaporative wind. A functional form for \mathcal{W} is

$$\mathcal{W} = \begin{cases} 0, & r < r_g, \\ \mathcal{W}_0 (r/r_g)^{-5/2}, & r \geq r_g, \end{cases}$$

for \mathcal{W}_0 a constant $\sim 10^{-12}$ g cm $^{-2}$ s $^{-1}$.

Show that the total mass loss rate from the disk due to the wind is $\dot{M}_w = 4\pi \mathcal{W}_0 r_g^2$. Hence give an estimate for the dispersal time, in years, of a protosolar nebula with initial mass $10^{-3} M_\odot$.

(c) Consider a steady state scenario in which mass is fed into the disk at a radius $\gg r_g$ at a rate $\dot{M}_d \geq \dot{M}_w$. Obtain an expression for \mathcal{F}_{acc} in terms of \dot{M}_d , \dot{M}_w , and r_g , then sketch its form as a function of r . Give a physical reason for its shape.

Compute the corresponding surface density profile Σ for fixed ν , assuming there is no viscous stress at the surface of the star (which we may assume is at $r = 0$). Sketch the form of Σ for the special case $\dot{M}_d = \dot{M}_w$.

Discuss what might happen to the inner disk at late times when the wind mass-loss rate approaches the accretion rate.

2 Gravitational instability in a layer of dust

The dynamics of a 2D layer of dust in a gaseous accretion disk can be modelled in the shearing sheet approximation via the following equations

$$\begin{aligned}\partial_t \sigma + \mathbf{v} \cdot \nabla \sigma &= -\sigma \nabla \cdot \mathbf{v}, \\ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + 2\Omega \mathbf{e}_z \times \mathbf{v} &= -\nabla \Phi_t - \nabla \Phi_{sg} - \frac{1}{\sigma} \nabla P - \epsilon \Omega (\mathbf{v} - \mathbf{U}_g).\end{aligned}$$

Here σ , \mathbf{v} , P , and Φ_{sg} denote the surface density, 2D velocity, vertically integrated pressure, and gravitational potential of the dust, while $\Phi_t = -(3/2)\Omega^2 x^2$ is the tidal potential, $\mathbf{U}_g = (-3/2)\Omega x \mathbf{e}_y$ is the velocity of the disk's gas, and ϵ measures the strength of the gas drag. We assume that the dust does not alter the motion of the gas. Finally, Φ_{sg} can be obtained from Poisson's equation

$$\nabla^2 \Phi_{sg} = 4\pi G \sigma \delta(z),$$

and the dust pressure from $P = c^2 \sigma$, where c is the rms speed of the dust particles.

(a) Consider small axisymmetric perturbations σ' , \mathbf{v}' , Φ'_{sg} around the equilibrium $\mathbf{v} = \mathbf{U}_g$ with $\sigma = \sigma_0$ a constant. Assuming the perturbations are $\propto e^{ikx+st}$, derive the following dispersion relation:

$$s^3 + 2s^2 \epsilon \Omega + (\bar{\omega}^2 + \epsilon^2 \Omega^2)s + \epsilon \Omega (\bar{\omega}^2 - \Omega^2) = 0,$$

where

$$\bar{\omega}^2 = \Omega^2 - 2\pi G \sigma_0 |k| + c^2 k^2.$$

You may assume that for such disturbances, $\Phi'_{sg} = -(2\pi G/|k|) \sigma'$.

(b) Show that when the dust particles decouple from the gas ($\epsilon = 0$), the dust is unstable when $\bar{\omega}^2 < 0$. Confirm that instability is assured when $Q < 1$, where $Q = c\Omega/(\pi G \sigma_0)$.

(c) Suppose $\bar{\omega}^2 > 0$ and that the particles are only weakly coupled to the gas, i.e. $0 < \epsilon \ll 1$. By making an expansion $s = s_0 + s_1 \epsilon + \dots$, show that there exist two density waves that are weakly damped.

(d) For small ϵ , the third 'secular' mode has an expansion $s = s_1 \epsilon + \mathcal{O}(\epsilon^2)$. Show that this mode grows when $\bar{\omega}^2 < \Omega^2$. Hence demonstrate that the dust is always unstable for sufficiently small k .

(e) If the dust layer has radial extent L , derive the rough instability criterion $Q \lesssim L/H$, where $H = c/\Omega$ is the scale height.

(f) Suppose the background fluid flow is turbulent. How do you think this will alter the onset of instability?

3 The vertically stratified MRI

A fully ionised gas in the vertically stratified shearing sheet obeys the momentum and induction equations,

$$\begin{aligned}\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + 2\Omega \mathbf{e}_z \times \mathbf{v} &= -\nabla \Phi_t - \frac{1}{\rho} \nabla \Psi + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi\rho}, \\ \partial_t \mathbf{B} + \mathbf{v} \cdot \nabla \mathbf{B} &= \mathbf{B} \cdot \nabla \mathbf{v} - (\nabla \cdot \mathbf{v}) \mathbf{B},\end{aligned}$$

where $\Phi_t = -(3/2)\Omega^2 x^2 + (1/2)\Omega^2 z^2$ is the tidal potential and Ψ is the combined gas and magnetic pressure. Suppose the box is penetrated by a uniform magnetic field $B_0 \mathbf{e}_z$, and exhibits a vertical density profile of $\rho = \rho_0 h$, where ρ_0 is the midplane density and $h = h(z/H)$ is a dimensionless function and H is the disc scale height. The equilibrium velocity is $\mathbf{v}_0 = -(3/2)\Omega x \mathbf{e}_y$.

The equilibrium state is perturbed by a disturbance of the following form

$$\begin{aligned}\mathbf{v}' &= e^{st} F (v'_x, v'_y, 0), \\ \mathbf{B}' &= B_0 e^{st} \left(\frac{1}{ik} \frac{dF}{dz} \right) (b'_x, b'_y, 0),\end{aligned}$$

where $v'_x, v'_y, b'_x,$ and b'_y are (complex) constants, $F = F(z/H)$ is a (real) dimensionless function, bounded on the surface of the disk, and k is a (real) wavenumber. We do not perturb the density or pressure.

(a) By determining the solvability condition for the horizontal components of the linearised momentum and induction equation, show that if F satisfies

$$\frac{d^2 F}{dz^2} + k^2 h F = 0 \quad (**)$$

then we recover the incompressible MRI dispersion relation

$$s^4 + (\Omega^2 + 2v_A^2 k^2) s^2 + v_A^2 k^2 (v_A^2 k^2 - 3\Omega^2) = 0,$$

where $v_A = B_0 / \sqrt{4\pi\rho_0}$.

(b) Suppose the vertical structure of the disk is given by $h = \text{sech}^2(z/H)$. Verify that Eq. (**) admits discrete solutions

$$F_n = P_n[\tanh(z/H)], \quad k_n = \sqrt{n^2 + n}/H,$$

where P_n is a Legendre polynomial of integer order n . Hence show that the MRI is stabilised when $\beta < 4/3$, where $\beta = 2\Omega^2 H^2 / v_A^2$. Give a physical explanation for why the disk is stabilised for sufficiently strong magnetic fields.

(c) If $\beta = 24$ what is n for the fastest growing mode?

[*Hint: The Legendre equation is*

$$\frac{d}{d\xi} \left[(1 - \xi^2) \frac{dF}{d\xi} \right] + \lambda F = 0$$

and admits solutions that are bounded at $\xi = \pm 1$ if $\lambda = n(n+1)$. These solutions are the Legendre polynomials, P_n .]

END OF PAPER