UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 3 June, 2014 1:30 pm to 4:30 pm

PAPER 55

STRUCTURE AND EVOLUTION OF STARS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Derive the Lane–Emden equation for a polytrope of index n.

What are the appropriate boundary conditions to solve this equation to produce a stellar model?

Solve the Lane–Emden equation for $n = 0$, as well as for $n = 1$. In each case find the polytropic radius.

Work out moments of inertia for stars of mass M and radius R that are polytropes of $n = 0$ and $n = 1$.

Find the equation relating the mass to the radius for a polytrope of index n in the form $M = AR^{p(n)}$, and give A in terms of K and n.

2

Explain what is meant by homologous models of stars.

For a set of fully radiative stars the opacity is $\kappa \propto \varrho^{\lambda} T^{\nu}$ and the energy generation rate is $\epsilon \propto \varrho T^{\eta}$, where λ, ν and η are constants and the stellar material behaves as a perfect gas with mean molecular weight μ. Find the mass-radius relationship for such stars. Assume for your calculations $M'_r = mM_r$, $T' = tT$, $P' = pP$, $L'_r = lL_r$, $R' = \xi r$.

Construct the Hertzsprung–Russell diagram for low-mass main-sequence stars burning hydrogen by the pp chain and with Kramers' opacity, and for higher mass stars burning hydrogen by the CNO cycle and with electron scattering opacity. In each case give the slope of $\log L/\log T_{\mathrm{eff}}$ relation.

3

A simple and fairly realistic model of main-sequence stars, called the "standard model", was devised by Eddington. This model is based on the following assumptions:

(a) the star is in radiative equilibrium (that is to say it is convectively stable);

(b) the stellar material behaves as a perfect gas providing pressure P_g along with radiation pressure P_r;

(c) the star is chemically homogeneous, so that the mean molecular weight μ is independent of radius;

(d) the opacity κ satisfies the relation

$$\kappa = \alpha \frac{M_r}{L_r} \frac{L}{M}$$

where α is a constant, and M_r, L_r are the interior mass and luminosity at radius r. This assumption is obviously artificial but happens to be reasonably well satisfied by many stars including the Sun.

In the standard model prove that:

(i) the ratio of gas pressure to the total (gas + radiation) pressure is a constant, β, independent of radius;

(ii) the luminosity cannot exceed $L = \frac{4\pi c G M}{\alpha}$, the Eddington limit;

(iii) the star is a polytrope with polytropic index $n = 3$;

(iv) the stellar mass $M \propto \frac{(1-\beta)^{1/2}}{\beta^2}$.

4

Write an essay on the Equation of State that describes matter in stellar interiors. You should include a discussion of the transition regions between various dominant contributions and present them on a $\log \varrho / \log T$ diagram.

END OF PAPER