UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 3 June, 2014 9:00 am to 12:00 pm

PAPER 54

ASTROPHYSICAL FLUID DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

$$\nabla^2 \Phi = 4\pi G \rho$$

You may assume that for any vectors $\mathbf{C}$ *and* $\mathbf{D}$

$\nabla \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{D} \cdot \nabla \times \mathbf{C} - \mathbf{C} \cdot \nabla \times \mathbf{D}$ *and* $\nabla \times (\mathbf{C} \times \mathbf{D}) = -\mathbf{D} \nabla \cdot \mathbf{C} + \mathbf{C} \nabla \cdot \mathbf{D} - \mathbf{C} \cdot \nabla \mathbf{D} + \mathbf{D} \cdot \nabla \mathbf{C}$.

For $\mathbf{u} = (u_R, u_\phi, u_z)$ *in cylindrical coordinates* (R, ϕ, z), *the components of* $\mathbf{u} \cdot \nabla \mathbf{u}$ *are:*

$$\left(u_R \frac{\partial u_R}{\partial R} + \frac{u_\phi}{R} \frac{\partial u_R}{\partial \phi} + u_z \frac{\partial u_R}{\partial z} - \frac{u_\phi^2}{R},\ \frac{u_R}{R} \frac{\partial (R u_\phi)}{\partial R} + \frac{u_\phi}{R} \frac{\partial u_\phi}{\partial \phi} + u_z \frac{\partial u_\phi}{\partial z},\ u_R \frac{\partial u_z}{\partial R} + \frac{u_\phi}{R} \frac{\partial u_z}{\partial \phi} + u_z \frac{\partial u_z}{\partial z} \right),$$

$$\nabla \times \mathbf{u} = \left(\frac{1}{R} \frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z}, \frac{\partial u_R}{\partial z} - \frac{\partial u_z}{\partial R}, \frac{1}{R} \left(\frac{\partial (R u_\phi)}{\partial R} - \frac{\partial u_R}{\partial \phi} \right) \right)$$

and

$$\nabla \cdot \mathbf{u} = \frac{1}{R} \frac{\partial (R u_R)}{\partial R} + \frac{1}{R} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z}$$

The symbols that appear in these equations may also appear in the questions without further definition.

1

a) Derive the conservation of energy equation for magnetized fluid moving under the ideal MHD equations in a fixed gravitational field in the form

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{F} = 0,$$

where the energy density is given by

$$\mathcal{E} = \rho \left(\frac{1}{2}|\mathbf{u}|^2 + \Phi \right) + \frac{p}{\gamma - 1} + \frac{|\mathbf{B}|^2}{2\mu_0}$$

and the energy flux is given by

$$\mathbf{F} = \rho \mathbf{u} \left(\frac{1}{2}|\mathbf{u}|^2 + \Phi + \frac{\gamma p}{\rho(\gamma - 1)} \right) - \frac{(\mathbf{u} \times \mathbf{B}) \times \mathbf{B}}{\mu_0}.$$

b) The fluid is axisymmetric and such that the magnetic field may be written, adopting cylindrical coordinates (R, ϕ, z), in the form

$$\mathbf{B} = \mathbf{B}_p + B_\phi \mathbf{e}_\phi,$$

where $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction. The density ρ and the poloidal field $\mathbf{B}_p$ are independent of time. The velocity is given by $\mathbf{u} = \mathbf{e}_\phi u_\phi$. The azimuthal component of velocity u_ϕ and the toroidal component of the magnetic field B_ϕ are functions of R and z and time. Show that the azimuthal components of the induction equation and the equation of motion respectively give

$$\frac{\partial B_\phi}{\partial t} = R\mathbf{B}_p \cdot \nabla \left(\frac{u_\phi}{R} \right) \quad \text{and}$$

$$\rho R \mu_0 \frac{\partial u_\phi}{\partial t} = \nabla \cdot (R\mathbf{B}_p B_\phi).$$

c) Show that the ratio of the poloidal component of the energy flux to the poloidal component of the angular momentum flux for these motions is equal to $\Omega = u_\phi/R$. Explain what condition $\Omega_0(R, z)$ should satisfy in order that $\Omega = \Omega_0$ be a steady state solution. Show further that Ω satisfies the equation

$$\rho R^2 \mu_0 \frac{\partial^2 \Omega}{\partial t^2} = \mathbf{B}_p \cdot \nabla \left(R^2 \mathbf{B}_p \cdot \nabla \Omega \right).$$

d) Now consider solutions of the form $\Omega = \Omega_0 + v \exp(\mathrm{i}(\mathbf{k} \cdot \mathbf{r} - \omega t))$, where $\mathbf{k}$ is the wavenumber, ω is the wave angular frequency, and v is a complex wave amplitude. Assuming that $|\mathbf{k}|$ and $|\omega|$ are very large so that the variation of all quantities other than the exponentials can be neglected, derive the local dispersion relation in the form

$$\omega^2 = \frac{(\mathbf{k} \cdot \mathbf{B}_p)^2}{\mu_0 \rho}.$$

2

a) Starting from the equations of ideal magnetohydrodynamics, show that the equation of motion of a self-gravitating gas with no external gravitational sources, moving under a magnetic field $\mathbf{B}$ and its pressure p, may be written in the form

$$\rho \frac{Du_i}{Dt} = \frac{\partial T_{ij}}{\partial x_j},$$

where the summation convention has been used and $T_{ij} = W_{ij} + M_{ij}$ for $i, j = 1, 2, 3$ are the components of a symmetric stress tensor with

$$M_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{|\mathbf{B}|^2}{2}\delta_{ij}\right), \text{ and } W_{ij} = -p\delta_{ij} - \frac{1}{4\pi G}\left(g_i g_j - \frac{|\mathbf{g}|^2}{2}\delta_{ij}\right).$$

The components of $\mathbf{g}$ are $g_i = -\partial\Phi/\partial x_i$ and the Cartesian coordinates $(x_1, x_2, x_3) \equiv (x, y, z)$.

b) Prove the tensor virial theorem in the form

$$\begin{aligned}\frac{1}{2}\frac{d^2 I_{ij}}{dt^2} &= 2K_{ij} - \mathcal{T}_{ij}, \quad \text{where} \quad I_{ij} = \int x_i x_j \rho d\tau, \\ K_{ij} &= \int \frac{\rho}{2} u_i u_j d\tau \quad \text{and} \quad \mathcal{T}_{ij} = \int T_{ij} d\tau.\end{aligned}$$

Here it is assumed that the magnitudes of the components of T_{ij} vanish sufficiently rapidly at large distances that surface contributions become negligible as the domain of integration is extended to the whole of space.

c) A cold axisymmetric magnetised gaseous configuration for which the pressure is negligible is at rest. Show that

$$\frac{1}{2}\frac{d^2 I_\perp}{dt^2} = \frac{1}{4\pi G}\int \left(\frac{4\pi G B_z^2}{\mu_0} - \left(\frac{\partial \Phi}{\partial z}\right)^2\right) d\tau, \tag{1}$$

$$\text{where } I_\perp = \int (x^2 + y^2)\rho d\tau.$$

d) Suppose now the matter is contained within an infinitesimally thin disk with $z = 0$ being the midplane and which extends to infinity horizontally. The contributions to the integral in (1) then come from the current-free vacuum regions exterior to the disk, with the regions above and below giving equal contributions on account of symmetry. By integrating Poisson's equation through the disk, show that on $z = 0_+$, $\partial\Phi/\partial z = 2\pi G\Sigma$, where it assumed that Φ, which satisfies Laplace's equation, is an even function of z and the surface density $\Sigma = \int_{-\infty}^{\infty} \rho dz$.

Show further that in the vacuum region above the disk, there is a scaled magnetostatic potential, Φ_M, such that $\sqrt{4\pi G/\mu_0}\,\mathbf{B} = -\nabla\Phi_M$, which also satisfies Laplace's equation, $\nabla^2\Phi_M = 0$, subject to the condition $\partial\Phi_M/\partial z = -\sqrt{4\pi G/\mu_0}B_{z0}$ on $z = 0_+$, with B_{z0} being the vertical component of the magnetic field in the disk midplane. Thus Φ_M is the gravitational potential associated with a surface density $-B_{z0}/\sqrt{\pi G \mu_0}$.

e) Assume that $B_{z0}^2 < \pi G \mu_0 \Sigma^2$ everywhere in the disk. By comparing the magnitudes of the vertical component of the gravitational force per unit mass arising from the two surface density distributions indicated above at an arbitrary point above the disk, show that the integral in (1) is negative and hence that the disk starts to collapse towards its centre.

3

a) Write down the equations governing the steady, spherically symmetric flow of a polytropic gas for which $p = K\rho^\gamma$, where K and $\gamma > 1$ are constants, in the gravitational potential due to a point mass $\Phi = -GM/r$. Show that the radial velocity, u_r, satisfies the equation

$$\frac{\left(u_r^2 - c_s^2\right)}{u_r}\frac{du_r}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2},$$

where

$$c_s^2 = \frac{\gamma p}{\rho}.$$

What is meant by the statement that there is a critical point and what are the conditions that should be satisfied there?

Show also that

$$\frac{1}{2}u_r^2 + \frac{\gamma p}{(\gamma - 1)\rho} - \frac{GM}{r} = C,$$

where C is a constant of integration.

b) A star has spherically symmetric wind for which the mass loss rate is $\dot{M}$. It satisfies the boundary conditions that as $r \to \infty$, $\rho \to 0$ and $u_r^2 \to 2C$. Define y such that $y = \rho r^\beta$, where $\beta = 4/(\gamma + 1)$. Show that y can be obtained from the equation

$$F(r) = r^\alpha \left(C + \frac{GM}{r}\right) = \frac{\gamma K y^{\gamma - 1}}{(\gamma - 1)} + \frac{\dot{M}^2}{32\pi^2 y^2} = G(y),$$

where $\alpha = 4(\gamma - 1)/(\gamma + 1)$. Find the condition that $F(r)$ has a minimum and show that when it does this occurs at $r = r_c = (GM/C)((5 - 3\gamma)/4(\gamma - 1))$. Show that this is also the radius of the critical point. Explain why the minima of $F(r)$ and $G(y)$ should coincide and have the same value in order to obtain a regular solution that passes through $r = r_c$.

By applying this condition, show that the mass loss rate is given by

$$\dot{M} = 4\pi r_c^2 (\gamma K)^{-1/(\gamma - 1)} \left(\frac{2C(\gamma - 1)}{5 - 3\gamma}\right)^{(\gamma + 1)/(2(\gamma - 1))}.$$

4

a) A non-magnetic star has a barotropic equilibrium state for which $p = p(\rho)$. As viewed in a cylindrical coordinate system (R, ϕ, z), the star is steady, axisymmetric and rotates non-uniformly about the z axis with angular velocity $\Omega(R)$. Thus $\mathbf{u} = \mathbf{u}_0 = (0, R\Omega(R), 0)$. Show that p and ρ are functions only of the quantity Ψ, where

$$\Psi = \Phi - \int_0^R R'\Omega^2(R')dR'.$$

b) The star undergoes small-amplitude axisymmetric adiabatic perturbations with constant γ, such that the velocity becomes $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$, where the velocity perturbation $\mathbf{u}'$ is of the form

$$\mathbf{u}' = (-\mathrm{i}\omega\xi_R(R,z), v_\phi(R,z), -\mathrm{i}\omega\xi_z(R,z)) \ \exp(-\mathrm{i}\omega t).$$

The perturbation of the gravitational potential may be neglected (Cowling approximation). Show that the azimuthal component of the linearized equation of motion gives

$$v_\phi + (\xi_R/R)d(R^2\Omega)/dR = 0.$$

Show that the linearization of the R and z components of the equation of motion gives

$$-\omega^2\boldsymbol{\xi} = \mathcal{F}\boldsymbol{\xi} = \frac{\delta\rho}{\rho^2}\nabla p - \frac{1}{\rho}\nabla\delta p - \kappa^2\xi_R\mathbf{e}_R,$$

$$\delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}), \qquad \delta p = -\boldsymbol{\xi}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\xi}.$$

Here $\kappa^2 = (2\Omega/R)(d(R^2\Omega)/dR)$, $\boldsymbol{\xi} = (\xi_R, \xi_z)$ and $\mathbf{e}_R$ is the unit vector in the radial direction.

c) Show that the force operator $\mathcal{F}$ is self-adjoint with respect to the inner product

$$\langle\boldsymbol{\eta}, \boldsymbol{\xi}\rangle = \int \rho\boldsymbol{\eta}^*\cdot\boldsymbol{\xi}d\tau.$$

Here the integral is taken over the volume occupied by the star which is assumed to have a bounding surface on which p and ρ vanish. Hence show that

$$\omega^2\int\rho|\boldsymbol{\xi}|^2 d\tau = -\langle\boldsymbol{\xi}, \mathcal{F}\boldsymbol{\xi}\rangle = \int\left(\frac{|\delta p|^2}{\gamma p} + \rho\mathcal{N}^2|\boldsymbol{\xi}\cdot\nabla\Psi|^2 + \rho\kappa^2|\xi_R|^2\right)d\tau,$$

where $\mathcal{N}$ is defined through $\mathcal{N}^2 = -\dfrac{1}{\rho}\dfrac{dp}{d\Psi}\left(\dfrac{1}{\gamma}\dfrac{d\ln p}{d\Psi} - \dfrac{d\ln\rho}{d\Psi}\right)$.

Explain, giving relevant criteria, how the above expression can be used, with the help of appropriate trial functions, to determine whether the system is stable or unstable.

d) Deduce that, if both $\mathcal{N}^2 > 0$ and $d(R^2\Omega)/dR > 0$ everywhere in the star, it is stable.

END OF PAPER