UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 6 June, 2014 9:00 am to 11:00 am

PAPER 53

ADVANCED COSMOLOGY

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) In the 3+1 formalism, we split spacetime using the line element

$$ds^2 = N^2 dt^2 - {}^{(3)}g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,,$$

with lapse function $N(t, x^i)$, shift vector $N^i(t, x^i)$ and the three-metric ${}^{(3)}g_{ij}(t, x^i)$ on constant time spacelike hypersurfaces Σ (Latin indices vary over 1,2,3). With normal vector n^μ to Σ satisfying $g_{\mu\nu}n^\mu n^\nu = 1$, we can define the projection operator by the expression $P^\mu{}_\alpha = \delta^\mu{}_\alpha - n^\mu n_\alpha$.

(i) Given the metric $g_{\mu\nu}$ defined by the line element above, find $g^{\mu\nu}(t, x^i)$ such that $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu{}_\nu$. For a future pointing $n_\mu = (N, 0, 0, 0)$, find n^μ. Show that the projected projection operator is itself, i.e., $P^\mu{}_\alpha\, P^\alpha{}_\nu = P^\mu{}_\nu$.

(ii) The derivative operator $\mathcal{D}_i$ on Σ can be defined by projecting the 3+1 covariant derivative ∇_μ as $\mathcal{D}_i = P^\mu_i \nabla_\mu$. Show that $\mathcal{D}_i\left({}^{(3)}g_{ij}\right) = 0$, where we note that the induced metric ${}^{(3)}g_{ij}$ on Σ can be represented as ${}^{(3)}g_{\alpha\beta} = P^\mu{}_\alpha\, P^\nu{}_\beta\, g_{\mu\nu} = g_{\alpha\beta} + n_\alpha n_\beta$.

(iii) The extrinsic curvature of Σ is defined as $K_{\alpha\beta} = P^\mu{}_\alpha\, P^\nu{}_\beta\, \nabla_\mu n_\nu$. Hence, or otherwise, show that

$$P^\mu{}_\lambda P^\nu{}_\sigma\, \nabla_\nu \left(P^\lambda{}_\mu\right) = -K_{\lambda\sigma}\, n^\lambda\,.$$

(b) According to the in-in formalism during inflation, the leading order correction to an operator Q is given by the expectation value

$$\langle Q(t)\rangle = \mathcal{R}e \left\langle -2iQ^I(t) \int_{-\infty(1-i\mathcal{E})}^{t} H^I_{\rm int}(t')dt' \right\rangle\,, \qquad (\dagger)$$

where we will assume the interaction Hamiltonian $H^I_{\rm int}$ at third-order is

$$H^I_{\rm int} = -\int d^3x\; a^3 \epsilon^2\, \zeta \left(\tfrac{d\zeta}{dt}\right)^2\,, \qquad (\ddagger)$$

with slow-roll parameter ϵ (which you may assume is effectively constant) and scale factor given by $a \approx -1/(H\tau)$ with Hubble constant H and conformal time τ (i.e., $dt = a d\tau$). Here, in the interaction picture, the linear density perturbation ζ is a Gaussian random field with power spectrum,

$$\langle \zeta^I(\mathbf{k}, \tau)\zeta^I(\mathbf{k}', \tau)\rangle = (2\pi)^3 u_{\mathbf{k}}(\tau)\, u^*_{\mathbf{k}}(\tau)\, \delta(\mathbf{k} + \mathbf{k}')\,, \qquad (*)$$

where the mode functions $u_{\mathbf{k}}(\tau)$ and their conformal time derivatives are given by

$$u_{\mathbf{k}}(\tau) = \frac{H}{\sqrt{4\epsilon M^2_{\rm Pl} k^3}}\,(1 + ik\tau)\, e^{-ik\tau}\,, \qquad u'_{\mathbf{k}}(\tau) = \frac{H}{\sqrt{4\epsilon M^2_{\rm Pl} k^3}} k^2 \tau e^{-ik\tau}\,.$$

(i) Briefly explain Wick's Theorem for a Gaussian random field. Use Wick's theorem, together with the power spectrum $(*)$ and the in-in formalism expression $(\dagger)$, to show that the three point correlator of ζ for the interaction Hamiltonian (‡) reduces to the following terms,

$$\begin{aligned}\langle\zeta(\mathbf{k}_1,0),\zeta(\mathbf{k}_2,0),\zeta(\mathbf{k}_3,0)\rangle \;&=\; \mathcal{R}e\biggl(-2i\int d\tau\int d^3p_1\,d^3p_2\,d^3p_3\\ &\times\;\frac{\epsilon^2}{(H\tau)^2}\,u_{\mathbf{k}_1}(0)\,u_{\mathbf{k}_2}(0)\,u_{\mathbf{k}_3}(0)\,u_{\mathbf{p}_1}(\tau)\,u'_{\mathbf{p}_2}(\tau)\,u'_{\mathbf{p}_3}(\tau)\\ &\times\;(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\left[\delta(\mathbf{k}_1+\mathbf{p}_1)\delta(\mathbf{k}_2+\mathbf{p}_2)\delta(\mathbf{k}_3+\mathbf{p}_3)+\text{cyclic perms.}\right]\biggr).\end{aligned}$$

(ii) Substitute the mode functions for the density field ζ and evaluate the integrals above explicitly to show that the three-point correlator becomes

$$\begin{aligned}\langle\zeta(\mathbf{k}_1,0),\zeta(\mathbf{k}_2,0),\zeta(\mathbf{k}_3,0)\rangle &= (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{H^4}{16\epsilon M_{\rm Pl}^4}\\ &\times\;\frac{1}{(k_1k_2k_3)^3}\left(\frac{k_2^2k_3^2}{K}+\frac{k_1k_2^2k_3^2}{K^2}+\text{cyclic perms.}\right).\end{aligned}$$

Explain how to take the appropriate limit as $\tau\to-\infty$. Briefly comment on whether the non-Gaussian parameter $f_{\rm NL}$ is expected to be detectable.

2

(i) What is meant by the one-particle distribution function in relativistic kinetic theory?

Show that the stress–energy tensor of a gas of photons is

$$T^{\mu\nu} = \int \frac{d^3\mathbf{p}}{E(\mathbf{p})} f p^\mu p^\nu \,,$$

where $\mathbf{p}$ is the 3-momentum and $E(\mathbf{p}) = |\mathbf{p}|$ is the energy of a photon defined with respect to an orthonormal tetrad, p^μ is the photon 4-momentum, and f is the one-particle distribution function.

(ii) In a linearly-perturbed Friedmann–Robertson–Walker model with scale factor $a(\eta)$, the perturbation to the distribution function can be written as

$$f(\eta, \mathbf{x}, \epsilon, \mathbf{e}) = \bar{f}(\epsilon)\left[1 - \Theta(\eta, \mathbf{x}, \mathbf{e})\frac{d\ln \bar{f}}{d\ln \epsilon}\right] .$$

Here, $\epsilon = aE$ is the comoving energy, $\mathbf{e}$ is the photon direction (with $\mathbf{p} = E\mathbf{e}$ and $\mathbf{e}^2 = 1$), and $\bar{f}(\epsilon)$ is the unperturbed distribution function. Conformal time is denoted by η and $\mathbf{x}$ is comoving position. Give a physical interpretation of $\Theta(\eta, \mathbf{x}, \mathbf{e})$.

For scalar perturbations, the Fourier transform of Θ is axisymmetric about the wavevector $\mathbf{k}$ and can be expanded in Legendre polynomials P_l as

$$\Theta(\eta, \mathbf{k}, \mathbf{e}) = \sum_{l\geqslant 0} (-i)^l \Theta_l(\eta, \mathbf{k}) P_l(\hat{\mathbf{k}}\cdot\mathbf{e}) \,,$$

where $\hat{\mathbf{k}}$ is a unit vector along $\mathbf{k}$. If the Fourier transform of the components of the stress–energy tensor on the orthonormal tetrad are written as (for $\mathbf{k} \neq 0$)

$$T^{\hat{0}\hat{0}} = \bar{\rho}_\gamma \delta_\gamma \,, \quad T^{\hat{0}\hat{\imath}} = \frac{4}{3}\bar{\rho}_\gamma i\hat{k}^i v_\gamma \,, \quad T^{\hat{\imath}\hat{\jmath}} = \frac{1}{3}\bar{\rho}_\gamma \left(\delta_\gamma \delta^{ij} + 4\hat{k}^{\langle i}\hat{k}^{j\rangle}\Pi_\gamma\right) \,,$$

where angle brackets around indices denote the symmetric, trace-free part, express the background energy density $\bar{\rho}_\gamma$ in terms of $\bar{f}(\epsilon)$ and show that the density contrast δ_γ and bulk velocity v_γ are given by

$$\delta_\gamma = 4\Theta_0 \,, \quad v_\gamma = -\Theta_1 \,.$$

In the rest of the question, you may assume that $\Pi_\gamma = -3\Theta_2/5$.

(iii) The Boltzmann hierarchy in the Newtonian gauge is (ignoring polarization)

$$\dot{\Theta}_l + k\left(\frac{l+1}{2l+3}\Theta_{l+1} - \frac{l}{2l-1}\Theta_{l-1}\right) = -\dot{\tau}\left[(\delta_{l0} - 1)\Theta_l - \delta_{l1} v_b + \frac{1}{10}\delta_{l2}\Theta_2\right] + \delta_{l0}\dot{\phi} + \delta_{l1} k\psi \,,$$

where v_b is the baryon velocity, ϕ and ψ are the metric potentials, and $\dot{\tau} = -a\bar{n}_e\sigma_{\mathrm{T}}$ is the differential optical depth (with $\bar{n}_e$ the unperturbed electron density). Overdots denote differentiation with respect to proper time and $k = |\mathbf{k}|$. Give the continuity and Euler equations for the time evolution of δ_γ and v_γ, respectively.

By considering conservation of total energy and momentum, show that the baryon density contrast δ_b and bulk velocity v_b evolve as

$$\dot{\delta}_b - kv_b - 3\dot{\phi} = 0\,, \quad \dot{v}_b + \frac{\dot{a}}{a}v_b + k\psi = -\frac{\dot{\tau}}{R}(v_\gamma - v_b)\,,$$

for some suitable R that you should specify. (Ignore the effects of baryon pressure.)

(iv) Explain what is meant by the tight-coupling approximation.

Consider scales small enough that the effects of gravity and cosmic expansion can be ignored. Show that to first-order in $k|\dot{\tau}|^{-1}$, the photon anisotropic stress

$$\Pi_\gamma \approx -\frac{4}{9}k\dot{\tau}^{-1}v_\gamma\,,$$

and that the effective Euler equation for the photon bulk velocity becomes

$$(1+R)\dot{v}_\gamma = -\frac{1}{4}k\delta_\gamma - \frac{1}{4}\left(\frac{R^2}{1+R} + \frac{8}{9}\right)k|\dot{\tau}^{-1}|\dot{\delta}_\gamma\,.$$

Hence show that, on small scales, δ_γ satisfies

$$\ddot{\delta}_\gamma + \frac{k^2|\dot{\tau}^{-1}|}{3(1+R)}\left(\frac{R^2}{1+R} + \frac{8}{9}\right)\dot{\delta}_\gamma + \frac{k^2}{3(1+R)}\delta_\gamma = 0\,.$$

Discuss, briefly, the form of the solutions of this equation for δ_γ.

[*Note that the continuity and Euler equations for scalar perturbations in the Newtonian gauge for a non-interacting component are*

$$\dot{\delta} - \left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(kv + 3\dot{\phi}) + 3\frac{\dot{a}}{a}\left(\frac{\delta P}{\bar{\rho}} - \frac{\bar{P}}{\bar{\rho}}\delta\right) = 0$$

$$\dot{v} + \frac{\dot{a}}{a}v + k\frac{\delta P}{\bar{\rho} + \bar{P}} + k\psi + \frac{2}{3}k\Pi + \frac{\dot{\bar{P}}}{\bar{\rho} + \bar{P}}v = 0\,,$$

where $\bar{P}$ is the unperturbed pressure, and δP is the perturbation.]

3

(i) Define the primordial bispectrum $B(k_1, k_2, k_3)$ and discuss briefly under what conditions viable inflation models can produce a measurable bispectrum. The local non-Gaussian model is constructed by adding the square of a Gaussian random field ζ_G to itself as

$$\zeta(\mathbf{x}) = \zeta_{\rm G}(\mathbf{x}) + \tfrac{3}{5} f_{\rm NL}[\zeta_{\rm G}^2(\mathbf{x}) - \langle \zeta_{\rm G}^2(\mathbf{x})\rangle] ,$$

where $f_{\rm NL}$ is the non-Gaussianity parameter. Show that the local-type bispectrum can be expressed as

$$B^{\rm loc}(k_1, k_2, k_3) = \tfrac{6}{5} f_{\rm NL}\big(P(k_1)P(k_2) + P(k_2)P(k_3) + P(k_3)P(k_1)\big) ,$$

given that the power spectrum $P(k)$ is defined through the relation

$$\langle \zeta_{\rm G}(\mathbf{k}_1)\zeta_{\rm G}(\mathbf{k}_2)\rangle = (2\pi)^3 P(k_1)\delta(\mathbf{k}_1 + \mathbf{k}_2) \quad \text{with} \quad k = |\mathbf{k}| ,$$

where the Fourier convention is $\zeta_G(\mathbf{k}) = \int d^3\mathbf{x}\, \zeta_G(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$.

(ii) If the non-Gaussianity is large enough, the observable angular bispectrum of the CMB can be linearly related to the primordial bispectrum. For scalar perturbations, we can write the spherical multipoles of the temperature anisotropy at time η_0 and position $\mathbf{x}_0$ in terms of the Fourier transform of the primordial curvature perturbation $\zeta(\mathbf{k})$ (or $\mathcal{R}(\mathbf{k})$) as

$$\Theta_{lm}(\eta_0, \mathbf{x}_0) = 4\pi(-i)^l \int \frac{d^3\mathbf{k}}{(2\pi)^3} g_l(\eta_0, k)\zeta(\mathbf{k}) Y^*_{lm}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}_0} .$$

Here, $\hat{\mathbf{k}} = \mathbf{k}/k$, the Y_{lm} are the spherical harmonics, and $g_l(\eta_0, k)$ is a transfer function. Show that the CMB 3-point function at time η_0 is

$$\langle \Theta_{l_1m_1}\Theta_{l_2m_2}\Theta_{l_3m_3}\rangle = b_{l_1l_2l_3}\mathcal{G}^{l_1l_2l_3}_{m_1m_2m_3} ,$$

where the reduced bispectrum

$$b_{l_1l_2l_3} = \frac{64}{(2\pi)^3}\int_0^\infty dx\, x^2 \int \prod_{i=1}^{3} \left[dk_i\, k_i^2 j_{l_i}(k_i x) g_{l_i}(\eta_0, k_i)\right] B(k_1, k_2, k_3) ,$$

and $\mathcal{G}^{l_1l_2l_3}_{m_1m_2m_3}$ is the (real-valued) Gaunt integral

$$\mathcal{G}^{l_1l_2l_3}_{m_1m_2m_3} \equiv \int d\hat{\mathbf{n}}\, Y_{l_1m_1} Y_{l_2m_2} Y_{l_3m_3} .$$

The j_l are the spherical Bessel functions.

(iii) Suppose that a given model predicts the shape of $b_{l_1l_2l_3}$ but not the amplitude $f_{\rm NL}$, so we can write

$$b_{l_1l_2l_3} = f_{\rm NL} b^{f_{\rm NL}=1}_{l_1l_2l_3} .$$

Full-sky, noise-free observations of the CMB temperature anisotropies are to be used to constrain $f_{\rm NL}$. Show that the estimator

$$\hat{f}_{\rm NL} = \frac{1}{6F}\sum_{l_1m_1}\sum_{l_2m_2}\sum_{l_3m_3} b^{f_{\rm NL}=1}_{l_1l_2l_3}\mathcal{G}^{l_1l_2l_3}_{m_1m_2m_3}\frac{\Theta_{l_1m_1}}{C_{l_1}}\frac{\Theta_{l_2m_2}}{C_{l_2}}\frac{\Theta_{l_3m_3}}{C_{l_3}}$$

is unbiased, i.e., the expectation value $\langle \hat{f}_{\rm NL} \rangle = f_{\rm NL}$, for a suitable normalisation F that you should determine.

Use Wick's theorem to show further that, in the Gaussian limit ($f_{\rm NL} = 0$), the cosmic variance of the estimator is $1/F$. You should assume that $\Theta_{00} = 0$.

[*You may use the following results: the plane-wave expansion is*

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{lm} i^l j_l(kx) Y^*_{lm}(\hat{\mathbf{x}}) Y_{lm}(\hat{\mathbf{k}})\,;$$

the addition theorem is

$$\sum_m Y_{lm}(\hat{\mathbf{n}}) Y^*_{lm}(\hat{\mathbf{n}}') = \frac{2l+1}{4\pi} P_l(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}')\,;$$

and the Gaunt integral satisfies

$$\sum_{m_1 m_2 m_3} \left(\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3}\right)^2 = \frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2 .]$$

END OF PAPER