UNIVERSITY OF CAMBRIDGE

## MATHEMATICAL TRIPOS Part III

Thursday, 5 June, 2014 1:30 pm to 4:30 pm

# PAPER 52

# BLACK HOLES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(a) Starting from the Schwarzshild spacetime in Schwarzschild coordinates, describe how to construct the Kruskal spacetime.

(b) Describe how to construct the Penrose diagram for Minkowski spacetime.

(c) Explain why it is expected that asymptotically flat initial data containing a trapped surface will satisfy the inequality

$$E \geqslant \sqrt{\frac{A_{\text{app}}}{16\pi}}$$

where $E$ is the ADM energy of the initial data and $A_{\text{app}}$ the area of the apparent horizon.

**2**

(a)(i) What is a null geodesic congruence? Let $U^a$ be tangent to the affinely parameterized geodesics of a null geodesic congruence. Let $B^a{}_b = \nabla_b U^a$. Explain why $U_a B^a{}_b = B^a{}_b U^b = 0$.

(ii) Explain how to construct a vector field $N^a$ such that (i) $N^2 = 0$ (ii) $U \cdot N = -1$ (iii) $U \cdot \nabla N^a = 0$

(iii) Let $P^a_b = \delta^a_b + U^a N_b + N^a U_b$ and $\hat{B}_{ab} = P^c_a P^d_b B_{cd}$. Explain how to define the expansion, rotation and shear of the congruence in terms of $\hat{B}_{ab}$.

(iv) Consider a null geodesic congruence containing the generators of a null hypersurface $\mathcal{N}$. Show that the rotation vanishes on $\mathcal{N}$.

(b) In ingoing Eddington–Finkelstein coordinates, the Reissner–Nordstrom metric is

$$ds^2 = -f dv^2 + 2dvdr + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \qquad f = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)$$

where $r_+ > r_- > 0$. Choose the time orientation so that $-\partial/\partial r$ is future-directed.

(i) Let $S$ denote the 2-sphere $v = v_0$, $r = r_0$. On $S$, let $X = -\partial/\partial r$. Show that $X^a$ is normal to $S$. Let $Y^a$ be another null vector normal to $S$ such that $X \cdot Y = -1$. Determine $Y^a$.

(ii) Calculate the expansion on $S$ of (1) the null geodesics with tangent vector $X^a$; (2) the null geodesics with tangent vector $Y^a$. [*Hint. For (1) choose $U^a = X^a$, $N^a = Y^a$ on $S$ and vice-versa in (2).*]

(iii) What is a trapped surface? Use your results to determine the range of values of $r_0$ for which $S$ is trapped.

(iv) State the Penrose singularity theorem and explain briefly how it applies to this spacetime.

**3**

(a) Consider an isolated uncharged star that undergoes gravitational collapse to form a black hole. Explain why it is believed that the spacetime at late time is characterized by only two parameters.

(b) In Boyer–Lindquist coordinates, the Kerr metric is

$$\begin{aligned} ds^2 &= -\frac{\left(\Delta - a^2\sin^2\theta\right)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{\left(r^2+a^2-\Delta\right)}{\Sigma}dtd\phi \\ &+ \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \end{aligned}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2$$

(i) For $M > a$, Kerr coordinates $(v, r, \theta, \chi)$ are defined by

$$dv = dt + \frac{r^2+a^2}{\Delta}dr \qquad d\chi = d\phi + \frac{a}{\Delta}dr$$

Determine the form of the Kerr metric in Kerr coordinates. Explain briefly why this metric can be analytically extended across the surface $r = r_+$ where $r_\pm = M \pm \sqrt{M^2 - a^2}$ (you may assume that $g \equiv \det g_{\mu\nu} \neq 0$ at $r = r_+$).

(ii) Let $k = \partial/\partial t$ and $m = \partial/\partial\phi$. Determine $k$ and $m$ in Kerr coordinates.

(iii) Show that $r = r_+$ is a Killing horizon of the Killing vector field $k + \Omega_H m$ for some constant $\Omega_H$ to be determined.

(iv) Explain the interpretation of $\Omega_H$.

(v) The wave equation can be written

$$0 = \nabla^a\nabla_a\Psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi\right)$$

Consider $\Psi$ of the form (in Boyer-Lindquist coordinates):

$$\Psi = e^{-i\omega t}e^{im\phi}R(r)\Theta(\theta)$$

Show that the wave equation reduces to ordinary differential equations for $R(r)$ and $\Theta(\theta)$. [*Hint. Show that the* $2\times 2$ *matrix corresponding to the* $t,\phi$ *components of* $g_{\mu\nu}$ *has determinant* $-\Delta\sin^2\theta$ *and hence determine the* $t,\phi$ *components of* $g^{\mu\nu}$.]

**4**

(a) Write an essay giving a detailed account of the quantum theory of a free scalar field in a globally hyperbolic spacetime. You should explain carefully why the particle concept is ambiguous in general and why it can be made unambiguous in a stationary spacetime. Describe how to calculate the expected number of particles produced in a spacetime that is stationary at early and late times but time-dependent in between.

(b) Explain why the discovery that black holes emit thermal radiation implies that the laws of black hole mechanics can be reinterpreted in thermodynamical terms.

## END OF PAPER