UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 5 June, 2014 9:00 am to 11:00 am

PAPER 51

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define a Poisson structure on an open set $U \subset \mathbb{R}^{2n}$ in terms of an anti-symmetric matrix $\omega^{ab} : U \to \mathbb{R}^{2n}$ where $a, b = 1, \ldots, 2n$, and show that the Jacobi identity implies

$$\sum_{d=1}^{2n} \left(\omega^{dc} \frac{\partial \omega^{ab}}{\partial x^d} + \omega^{db} \frac{\partial \omega^{ca}}{\partial x^d} + \omega^{da} \frac{\partial \omega^{bc}}{\partial x^d} \right) = 0.$$

Assume that ω^{ab} is invertible, and demonstrate that the corresponding symplectic two-form is closed as a consequence of the Jacobi identity.

2

Consider a Lie group G of upper triangular matrices of the form

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

and construct its Lie algebra $\mathfrak{g}$.

Construct explicitly the most general right invariant metric h on G and show that the isometry group of (G, h) contains G. Find expressions for three Killing vector fields generating $\mathfrak{g}$.

Now assume that h is diagonal in the basis of right-invariant one-forms and put h in the Kaluza–Klein form

$$h = V(dx + A)^2 + \gamma$$

with respect to the Killing vector $\partial/\partial x$, where the function V, the one-form A and the metric γ should be determined.

3

Write an essay on a Chern number and Chern–Simons three-form.

4

Consider a flat Euclidean metric on $\mathbb{R}^4$

$$g = dzd\bar{z} + dwd\bar{w}$$

with the volume form $dw \wedge dz \wedge d\bar{w} \wedge d\bar{z}$, where $(w, z) \in \mathbb{C}^2$.

Show that the space $\Lambda^2_+(\mathbb{R}^4)$ is spanned by real two-forms $\omega_1, \omega_2, \omega_3$, where

$$\omega_1 + i\omega_2 = dw \wedge dz, \quad \omega_3 = i(dw \wedge d\bar{w} + dz \wedge d\bar{z})$$

and deduce that in these coordinates the anti-self-dual Yang–Mills (ASDYM) equations are

$$F_{wz} = 0, \quad F_{w\bar{w}} + F_{z\bar{z}} = 0, \quad F_{\bar{w}\bar{z}} = 0,$$

where

$$F = dA + A \wedge A = \frac{1}{2} F_{ab}\, dy^a \wedge dy^b$$

is a real two-form with values in a Lie algebra $\mathfrak{g}$, and $y^a = (w, z, \bar{w}, \bar{z})$.

Use the first two equations to deduce the existence of a complex gauge such that

$$A_w = 0, \quad A_z = 0, \quad A_{\bar{w}} = \partial_z K, \quad A_{\bar{z}} = -\partial_w K$$

where $K = K(w, z, \bar{w}, \bar{z})$ is a $\mathfrak{g}$-valued function on $\mathbb{R}^4$ and thus reduce ASDYM to a single PDE for the function K.

END OF PAPER