UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 2 June, 2014 9:00 am to 12:00 pm

PAPER 50

GENERAL RELATIVITY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds of dimensions M and N, respectively. Let $\phi : \mathcal{M} \to \mathcal{N}$ be smooth.
(a) Briefly describe how (i) the pullback of a function f, (ii) the pushforward of a curve λ, (iii) the pushforward of a vector V, and (iv) the pullback of a covector η are defined.

(b) Now consider the special case where $M = N - 1$, so that the image $\Sigma := \phi[\mathcal{M}]$ is a hypersurface in $\mathcal{N}$. Let n be the unit normal field (assumed to be either timelike everywhere or spacelike everywhere) on Σ and $\perp$ the projector (in $\mathcal{N}$) onto Σ defined by

$$\perp^a{}_b = \delta^a{}_b \pm n^a n^b \,,$$

where the upper sign corresponds to the case that n is timelike and the lower sign corresponds to a spacelike n. Let $p \in \mathcal{M}$ be an arbitrary point.
i) Let V be a vector at p on $\mathcal{M}$. Show that the pushforward of V satisfies

$$\phi_* V = \perp(\phi_* V)\,.$$

ii) Let ω be a tensor of type $\binom{0}{s}$ on $\mathcal{N}$ for an integer $s \geqslant 1$. Show that the pullback of ω satisfies

$$\phi^* \omega = \phi^*(\perp\omega)\,.$$

iii) Let T be a tensor of type $\binom{r}{0}$ on $\mathcal{M}$ for an integer $r \geqslant 1$. Show that the pushforward of T satisfies

$$\phi_* T = \perp(\phi_* T)\,.$$

Here the projection of a tensor of arbitrary type $\binom{r}{s}$ is defined by

$$\perp T^{a_1\ldots a_r}{}_{b_1\ldots b_s} = \perp^{a_1}{}_{c_1} \ldots \perp^{a_r}{}_{c_r} \perp^{d_1}{}_{b_1} \ldots \perp^{d_s}{}_{b_s} T^{c_1\ldots c_r}{}_{d_1\ldots d_s}\,.$$

(c) Now consider the specific case where $\mathcal{N} = \mathbb{R}^3$ with Cartesian coordinates $y^\alpha \equiv (x,\, y,\, z)$ and Euclidean metric $g_{\alpha\beta} = \delta_{\alpha\beta}$, and $\mathcal{M}$ is the two-dimensional cylinder defined as the set of points in $\mathbb{R}^3$ with $x^2 + y^2 = \rho^2$, where $\rho > 0$ is a constant. Let $x^i \equiv (\varphi,\, \hat{z})$, $0 < \varphi < 2\pi$, $\hat{z} \in \mathbb{R}$ be coordinates on the cylinder (the need to introduce a second coordinate patch to complete an atlas can be ignored for the purposes of this exercise). Let $\phi : \mathcal{M} \to \mathcal{N}$ such that $p \in \mathcal{M}$ with coordinates $(\varphi,\, \hat{z})$ is mapped to $\phi(p) \in \mathcal{N}$ with coordinates

$$\begin{aligned} x &= \rho \cos\varphi\,, \\ y &= \rho \sin\varphi\,, \\ z &= \hat{z}\,. \end{aligned}$$

i) Calculate the pullback of the metric $g_{\alpha\beta}$. What is the Riemann tensor associated with the pullback of the metric (this does not require a calculation)?
ii) Calculate the trace of the extrinsic curvature on the cylinder. Note: specify the sign convention you choose for the extrinsic curvature.

2

In SI units, Newton's gravitational constant and the speed of light have (up to three significant digits) the values

$$G = 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}\,,$$
$$c = 3.00 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}\,.$$

Also in SI units, radius, mass and luminosity of the sun are given by

$$R_\odot = 6.96 \times 10^8 \text{ m}\,,$$
$$M_\odot = 1.99 \times 10^{30} \text{ kg}\,,$$
$$L_\odot = 3.85 \times 10^{26} \text{ W}\,,$$

where $1 \text{ W} = 1 \text{ kg}\,\text{m}^2/\text{s}^3$. In general relativity we often use units where $G = 1 = c$ which establishes a natural relation between the units of mass, time, and length.

(a) Using relativistic units where $G = 1 = c$, give an order-of-magnitude estimate for the mass of the sun expressed (i) in metres, (ii) in seconds. Give an order-of-magnitude estimate of the Newtonian gravitational potential at the surface of the sun.

(b) Give an order-of-magnitude estimate of the solar luminosity in relativistic units. Assuming the solar luminosity to be constant, how many years would it roughly take the sun to radiate its entire mass away?

(c) Give the definition of the quadrupole tensor I_{ij} of a localized (inside a volume $\mathcal{V}$) matter distribution. What is the quadrupole tensor for a set of N discrete point particles (i.e. their energy density is given by Dirac delta functions) of masses $m_{(A)}$ ($A = 1, \ldots, N$) at locations $\boldsymbol{y}_{(A)}$ moving at speeds small compared to the speed of light?

(d) Now consider 2 point particles of equal mass m that start from rest at $\pm z_0$ (and $x_0 = y_0 = 0$) and fall toward each other according to Newtonian gravity.

Using units where $G = 1 = c$, derive the equations of motion for the particles in the rest frame of the centre of mass and show that the only non-zero component of the third time derivative of the quadrupole tensor is given by

$$\dddot{I}_{zz} = \frac{m^2}{z^2}\sqrt{\frac{m}{2}}\sqrt{\frac{1}{z} - \frac{1}{z_0}}\,. \qquad (\dagger)$$

(e) The quadrupole formula gives the power of the radiated gravitational waves as

$$P = \frac{1}{5}\dddot{Q}_{ij}\dddot{Q}_{ij}\,,$$

where Q_{ij} is the traceless quadrupole tensor.

By using the result of Eq. ($\dagger$), calculate the power P as a function of z. Calculate the total radiated energy predicted by the quadrupole formula for two particles falling in from infinity, i.e. $z_0 \to \infty$, assuming that the particles stop radiating gravitational waves when $z = 2m$. Give an order of magnitude estimate of the time (in years) it would take the sun to radiate a similar fraction of its mass assuming a constant luminosity $L_\odot$ as given in question (b).

3

On a Lorentzian or Riemannian manifold, the Riemann curvature tensor in a coordinate basis x^α is given by

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}\,,$$

where $\Gamma^\alpha_{\beta\gamma}$ are the connection coefficients.

Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds of dimension m and $n > m$, respectively. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a smooth map, x^i, $i = 1, \ldots, m$ be a coordinate chart on $\mathcal{M}$, and y^α, $\alpha = 1, \ldots, n$ a coordinate chart on $\mathcal{N}$.

(a) Let $p \in \mathcal{M}$ be an arbitrary point. Show that in the corresponding coordinate bases $(\partial/\partial x^i)$ and $(\partial/\partial y^\alpha)$, the components V^i of a vector $V \in \mathcal{T}_p(\mathcal{M})$ are related to the components $(\phi_* V)^\alpha$ of its pushforward by

$$(\phi_* V)^\alpha = \frac{\partial y^\alpha}{\partial x^i} V^i\,.$$

(b) Show that the components of a covector $\omega \in \mathcal{T}^*_{\phi(p)}(\mathcal{N})$ are related to those of its pullback by

$$(\phi^* \omega)_i = \frac{\partial y^\alpha}{\partial x^i} \omega_\alpha\,.$$

(c) Consider the Hamiltonian and momentum constraints of the $3+1$ split of the Einstein equations in vacuum given by

$$\mathcal{R} + K^2 - K_{ij}K^{ij} = 0\,,$$
$$D_j K^j{}_i - D_i K = 0\,.$$

Here, $i, j = 1, 2, 3$ are spatial indices, K_{ij} is the extrinsic curvature, and $\mathcal{R}$ and D_i are the Ricci curvature scalar and the covariant derivative associated with the *Levi–Civita* connection of the three-dimensional induced metric γ_{ij}. Assume that the extrinsic curvature vanishes, $K_{ij} = 0$, and that

$$\gamma_{ij} = \psi^4 \delta_{ij}\,,$$

where δ_{ij} is the flat Euclidean metric diag(1, 1, 1) and the conformal factor $\psi \neq 0$ is an unknown function of the x^i.
Show that the constraint equations reduce to the flat-space Laplace equation for the conformal factor ψ: $\triangle \psi = 0$.

(d) For the case of asymptotically flat boundary conditions, i.e. $\lim_{r\to\infty} \psi = 1$, where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, find a non-trivial (i.e. non-constant) solution of the equation derived in (c) for the conformal factor ψ.

4

(a) Consider the Lagrangian of the geodesic equation

$$L = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,,$$

for timelike $dx^\mu/d\tau$, where $\mu = 0, \ldots, 3$, and τ is proper time along the geodesic. In the Newtonian limit, we consider small velocities and the spacetime metric is given by

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\delta_{ij}dx^i\,dx^j\,,$$

where x^i, $i = 1, \ldots, 3$, are spatial Cartesian coordinates, δ_{ij} is the Kronecker delta, Φ is a function $|\Phi(x^1, x^2, x^3)| \ll 1$ and we only consider velocities much smaller than the speed of light.

Show through variation of the Lagrangian L that the geodesic equation of motion reproduces the Newtonian equation of motion for a particle in a gravitational field.

(b) In the linearized approximation to general relativity, we consider the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,$$

where $\eta_{\mu\nu}$ is the Minkowskian metric in Cartesian coordinates $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, the components of h are small, $h_{\mu\nu} \ll 1$, but in contrast to the Newtonian limit, time derivatives are not neglected relative to spatial derivatives. In this case, the first order perturbation of the Ricci tensor is

$$\delta R_{\alpha\beta} = \frac{1}{2}\left(-\eta^{\mu\nu}\partial_\mu\partial_\nu h_{\alpha\beta} + \partial_\mu\partial_\alpha h^\mu{}_\beta + \partial_\mu\partial_\beta h^\mu{}_\alpha - \partial_\alpha\partial_\beta h\right)\,,$$

where $h \equiv h^\mu{}_\mu$ and $h^\mu{}_\nu \equiv \eta^{\mu\rho}h_{\rho\nu}$, i.e. indices in $h_{\mu\nu}$ are raised with the background metric.
i) Calculate the first order perturbation $\delta G_{\alpha\beta}$ of the Einstein tensor.
ii) Consider the Lagrangian

$$L = -\frac{1}{2}\left[(\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\right]\,.$$

Show that the Euler–Lagrange equation for varying the Lagrangian with respect to $h^{\alpha\beta}$ produces the vacuum Einstein equations at first order, $\delta G_{\alpha\beta} = 0$ with the $\delta G_{\alpha\beta}$ calculated above.

END OF PAPER