UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 4 June, 2014 9:00 am to 12:00 pm

PAPER 49

COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The metric for a homogenous and isotropic FRW spacetime is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \ ,$$

where $a(t)$ is the scale factor and k parameterises spatial curvature. The evolution of the scale factor is determined by the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i(a) - \frac{k}{a^2} \ ,$$

where $H = \dot{a}/a$ is the Hubble parameter and ρ_i denote the relevant energy densities (matter, radiation, etc.).

i) Consider a curved FRW universe filled with a single fluid with constant equation of state $w = P/\rho$.

Derive the evolution equation for the fluid density ρ from the first law of thermodynamics. Hence, or otherwise, show that

$$\frac{d\Omega_k}{d\ln a} = (1+3w)\Omega_k(1-\Omega_k) \ , \qquad \text{where} \qquad \Omega_k(a) \equiv -\frac{k}{(aH)^2} \ .$$

Use this result to explain the *flatness problem* of the standard Big Bang cosmology. [10]

ii) Now, consider a flat FRW universe with present radiation density $\Omega_{r,0} \equiv 8\pi G\rho_{r,0}/(3H_0^2)$ and matter density $\Omega_{m,0} = 1 - \Omega_{r,0}$.

Write the Friedmann equation in conformal time, $\mathrm{d}\tau = \mathrm{d}t/a(t)$, using the Hubble constant $H_0 \equiv H(\tau_0)$ and setting the scale factor today equal to unity, $a(\tau_0) \equiv 1$. Show that the equation is solved by

$$a(\tau) = A\tau^2 + B\tau \ ,$$

where the constants A and B should be determined.

For $\Omega_{r,0} \sim 10^{-4}$, calculate the angular scale subtended by the causal horizon at matter-radiation equality. Justify any approximations you make.

Given that recombination happens shortly after matter-radiation equality, explain the *horizon problem* of the standard Big Bang cosmology. [10]

2

The distribution function for massless neutrinos is

$$f_\nu(p) = \left[\exp\left(\frac{p - \mu_\nu}{T_\nu}\right) + 1\right]^{-1} ,$$

where T_ν is the neutrino temperature and μ_ν is the chemical potential. A neutrino species with $|\mu_\nu| \gg T_\nu$ is called *degenerate*.

i) Show that the energy density of degenerate neutrinos is

$$\rho_\nu \approx \frac{|\mu_\nu|^4}{8\pi^2} ,$$

where $\mu_\nu \propto T_\nu$ in thermal equilibrium. What is the contribution of the corresponding anti-neutrinos? [6]

ii) Derive a bound on μ_ν/T_ν from the requirement that the present energy density in degenerate neutrinos doesn't exceed the critical density, $\rho_{\rm crit} = 6 \times 10^3\, T_0^4$, where T_0 is the present CMB temperature. [8]

iii) Discuss *qualitatively* how degenerate neutrinos would affect the production of primordial helium. [6]

3

The position $q(t)$ of a one-dimensional harmonic oscillator has the following action

$$S = \frac{1}{2}\int \mathrm{d}t\,\left[\dot{q}^2 - \omega^2 q^2\right] \ ,$$

where the overdot denotes a derivative with respect to time t. The position operator can be written as

$$\hat{q}(t) = q(t)\hat{a} + q^*(t)\hat{a}^\dagger \ ,$$

where $q\dot{q}^* - \dot{q}q^* \equiv i$ and $[\hat{a}, \hat{a}^\dagger] = 1$ (in units where $\hbar \equiv 1$). The vacuum state $|0\rangle$ is defined through $\hat{a}|0\rangle = 0$. Write the mode function as $q(t) = r(t)e^{is(t)}$, where $r(t)$ and $s(t)$ are real functions of time.

Derive the form of the mode function for which $\langle 0|\hat{H}|0\rangle$ is minimised, where $\hat{H}$ is the Hamiltonian. [*Hint: Remember that the functions* $r(t)$ *and* $s(t)$ *are constrained by the normalisation of* $q(t)$.] Compute the vacuum expectation value $\langle|\hat{q}|^2\rangle \equiv \langle 0|\hat{q}^\dagger\hat{q}|0\rangle$. [10]

Now, consider the quadratic action for the comoving curvature perturbation $\mathcal{R}$ during slow-roll inflation

$$S = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x\, z^2\Big[(\mathcal{R}')^2 - (\partial_i\mathcal{R})^2\Big] \ , \qquad z^2 \equiv 2a^2\varepsilon \ ,$$

where the prime denotes a derivative with respect to conformal time τ, $a(\tau)$ is the scale factor of the FRW metric and $\varepsilon \equiv -\dot{H}/H^2$ is the slow-roll parameter.

Write the action in terms of the field $v \equiv z\mathcal{R}$ and derive the equation of motion for the Fourier mode $v_{\boldsymbol{k}}(\tau)$, for the case that the spacetime can be approximated by the de Sitter solution $a \approx -(H\tau)^{-1}$, with $H \approx$ *const.*

With reference to the first part of the question, explain the definition of the *Bunch-Davies (BD) vacuum.* Show that the mode function corresponding to the BD vacuum in de Sitter space is

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \ .$$

Determine the power spectrum of curvature perturbations in the superhorizon limit, $P_{\mathcal{R}}(k) \equiv \lim_{k\tau\to 0}|\mathcal{R}_k(\tau)|^2$. [10]

4

Consider scalar perturbations in a flat universe dominated by a perfect fluid with constant equation of state $w = P/\rho \geqslant 0$. The perturbed Robertson–Walker line element is

$$\mathrm{d}s^2 = a^2(\tau)\left[(1+2\Phi)\mathrm{d}\tau^2 - (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] ,$$

where τ is conformal time and $a(\tau) \propto \tau^{2/(1+3w)}$ is the scale factor. The linearized Einstein equations are

$$\begin{aligned} \nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) &= 4\pi G a^2 \bar{\rho}\delta , && (1) \\ \Phi' + \mathcal{H}\Phi &= -4\pi G a^2(\bar{\rho} + \bar{P})v , && (2) \\ \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi &= 4\pi G a^2\, \delta P , && (3) \end{aligned}$$

where primes denote derivatives with respect to τ, overbars denote homogeneous background quantities, $\delta \equiv \delta\rho/\rho$ is the fractional density perturbation, δP is the pressure perturbation, $\partial_i v$ is the peculiar velocity and $\mathcal{H} \equiv a'/a$.

Show that

$$\Phi'' + \frac{6(1+w)}{1+3w}\frac{1}{\tau}\Phi' - w\nabla^2\Phi = 0 .$$

Find the solution for a single Fourier mode, $\Phi(k,\tau)$, for the case of a radiation fluid ($w = \frac{1}{3}$) and for a pressureless matter fluid ($w = 0$). In each case, discuss the evolution when the mode is inside and outside the Hubble radius. [10]

[*Hint: You may use that the solutions to Bessel's equation,*

$$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{du}{dx} + \left(1 - \frac{2}{x^2}\right)u = 0 ,$$

are $j_1(x) = x^{-2}(\sin x - x\cos x)$ *and* $n_1(x) = -x^{-2}(\cos x + x\sin x)$.]

The real universe can be modelled as a mixture of radiation (r) and cold dark matter (c). Assume scale-invariant initial conditions for Φ.

i) Sketch the solutions $k^{3/2}\Phi(k,a)$ for three different Fourier modes: $k \gg \mathcal{H}_{\rm eq}$, $k \ll \mathcal{H}_{\rm eq}$ and $k \sim \mathcal{H}_{\rm eq}$, where $\mathcal{H}_{\rm eq} \equiv \mathcal{H}(\tau_{\rm eq})$ is the Hubble rate at matter-radiation equality. [2]

ii) In the matter-dominated era, determine how the growing mode of the dark matter density contrast δ_c evolves inside the Hubble radius. Sketch the solutions $k^{3/2}\,\delta_c(k,a)$ for the same three Fourier modes as in part i). [4]

iii) Sketch the dark matter power spectrum, $P_{\delta_c}(k,z) \equiv |\delta_c(k,z)|^2$, at redshift $z = 1$. Explain the k-scaling for $k > \mathcal{H}_{\rm eq}$ and $k < \mathcal{H}_{\rm eq}$. [4]

END OF PAPER