UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Tuesday, 10 June, 2014 1:30 pm to 3:30 pm

PAPER 47

CLASSICAL AND QUANTUM SOLITONS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

In this paper the sine-Gordon field theory is defined to be the theory of a real scalar field $\phi(x,t)$ with Lagrangian density,

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{\beta^2}\left[\cos\left(\beta\phi\right) - 1\right]$$

where β is a dimensionless coupling and m is a mass scale.

1

Write an essay on the classical integrability of sine-Gordon theory. Your account should include a definition of the Bäcklund transform, $\mathcal{B}_a[\phi]$, depending on a parameter a, of a sine-Gordon solution $\phi(x,t)$. You should show that $\phi_a(x,t) = \mathcal{B}_a[\phi]$ also obeys the sine-Gordon equation. You should discuss the application of the Bäcklund transform to finding an infinite tower of conserved charges and to constructing two-soliton solutions. You should also give a qualitative discussion of multi-soliton scattering and its relation to classical integrability.

For the purposes of this question you may assume the commutativity of the Bäcklund transform, $\mathcal{B}_a\mathcal{B}_b = \mathcal{B}_b\mathcal{B}_a$, and the consequent relation,

$$\tan\left(\frac{\phi_{a,b} - \phi_0}{4}\right) \;=\; \left(\frac{b+a}{b-a}\right)\tan\left(\frac{\phi_b - \phi_a}{4}\right)$$

between four solutions of the sine-Gordon equation: $\phi_0(x,t)$, $\phi_a(x,t) = \mathcal{B}_a[\phi_0]$, $\phi_b(x,t) = \mathcal{B}_b[\phi_0]$ and $\phi_{a,b}(x,t) = \mathcal{B}_a\mathcal{B}_b[\phi_0] = \mathcal{B}_b\mathcal{B}_a[\phi_0]$.

2

Consider a theory of two real scalar fields, $\phi(x,t)$ and $\alpha(x,t)$, in $1+1$ dimensions with Lagrangian density,

$$\mathcal{L} \;=\; \frac{r}{4}\left[\partial_\mu\phi\partial^\mu\phi \,+\, \sin^2\phi\,\left(\partial_\mu\alpha\partial^\mu\alpha \,-\, m^2\right)\right] \,-\, \frac{\theta}{4\pi}\epsilon^{\mu\nu}\partial_\mu\left(\cos\phi\right)\,\partial_\nu\alpha$$

where r and θ are dimensionless parameters, m is a mass scale and $\epsilon^{\mu\nu}$ is the antisymmetric tensor in two dimensional spacetime. The fields ϕ and α are each subject to periodic identification $\phi \sim \phi + 2\pi$ and $\alpha \sim \alpha + 2\pi$.

Field configurations with different boundary conditions for $\phi(x,t)$ at left and right spatial infinity are classified by a topological charge,

$$T \;=\; -\frac{1}{2}\left[\left.\cos\phi\right|_{x=+\infty} \,-\, \left.\cos\phi\right|_{x=-\infty}\right]$$

Perform the following calcuations with $\theta = 0$ except where otherwise stated.

i) Find a Bogomoln'yi lower bound for the energy of static configurations in terms of T.

ii) Find a static kink solution with $T = 1$ which saturates the bound.

The theory also has a global symmetry under shifts of α,

$$\alpha(x,t) \,\to\, \alpha(x,t) \,+\, c, \qquad\qquad \phi(x,t) \,\to\, \phi(x,t)$$

where $0 \leqslant c < 2\pi$ is an arbitrary constant.

iii) Derive the Noether charge Q corresponding to this global symmetry. How is your answer modified for non-zero θ?

iv) Find a time-dependent kink solution with $T = 1$ of the form,

$$\alpha \,=\, \omega t \qquad \phi \,=\, \phi(x)$$

calculate its mass M and global charge Q. Eliminate ω to obtain an expression for M in terms of Q and the parameters m and r.

v) Apply the Bohr-Sommerfeld quantization condition to the family of time-dependent solutions to find the corresponding semiclassical spectrum of particles in the theory, giving the allowed values of Q. How is the spectrum modified for non-zero θ?

3 The quantum spectrum of the sine-Gordon model includes a kink K and anti-kink $\bar{K}$ of equal mass $M = 8m/\gamma$ where γ is related to the coupling constant β appearing in the Lagrangian by,

$$\gamma = \frac{\beta^2}{1 - \beta^2/8\pi}$$

The theory also contains a tower of K-$\bar{K}$ bound states or breathers $\mathcal{B}_k$, for $k = 1, 2, \ldots, k_{max}$, whose mass increases monotonically with k.

For the special values of the coupling $\gamma = 8\pi/n$ where n is a positive integer, the exact transmission amplitude for Kink -anti-Kink (K-$\bar{K}$) scattering with rapidity difference θ is,

$$S_T(\theta) = e^{i\pi n} \prod_{k=1}^{n-1} \frac{e^{\theta - i(\pi k/n)} + 1}{e^{\theta} + e^{-i(\pi k/n)}}$$

and the exact reflection amplitude vanishes. For this value of the coupling, deduce the masses of the bound states $\mathcal{B}_k$ and their total number k_{max}.

In the following let $S_{k,l}(\theta)$ denote the transmission amplitude for scattering of two breathers $\mathcal{B}_k$ and $\mathcal{B}_l$ at rapidity difference θ.

For $\gamma = 8\pi/n$, the two-body scattering amplitude for the lightest breather state $\mathcal{B}_1$ is given by,

$$S_{1,1}(\theta) = \frac{\sinh(\theta) + i \sin\left(\frac{\pi}{n}\right)}{\sinh(\theta) - i \sin\left(\frac{\pi}{n}\right)}$$

Explain why it is not necessary to specify separate transmission and reflection amplitudes in this case. Show that the position of the pole in the above expression is consistent with the formation of $\mathcal{B}_2$ as an on-shell intermediate state in the s-channel (ie check that energy and momentum are conserved in such a process).

Describe the fusion procedure for calculating bound state S-matrices and give a heuristic explanation of its relation to integrability.

Apply the fusion procedure to determine the transmission amplitude $S_{2,1}(\theta)$.

Show that the pole of $S_{2,1}(\theta)$ nearest the real axis in the complex θ plane can be accounted for by the exchange of an on-shell one-particle state in the t-channel. You should identify the particle exchanged.

You may use the following trigonometric identity, which holds for any α and β, without proof;

$$\sin^2(\alpha) + \sin^2(\beta) - 2\sin(\alpha)\sin(\beta)\cos(\alpha - \beta) = \sin^2(\alpha - \beta).$$

END OF PAPER