UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 9:00 am to 12:00 pm

PAPER 46

STRING THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A constrained mechanics model with $2N$ phase space coordinates $\{q^I, p_I; I = 1, \ldots, N\}$ has the action

$$I[q,p;\lambda] = \int dt \left\{ \dot{q}^I p_I - \lambda^i \varphi_i(q,p) \right\} ,$$

where $\{\lambda^i; i = 1, \ldots, n < N\}$ are Lagrange multipliers for the constraints $\varphi_i = 0$. Use this action to explain what it means to say that constraints are "first-class". Given that they *are* first-class, under what condition is their Poisson bracket algebra a Lie algebra? Explain why first-class constraints imply gauge invariances of the action.

Write down the action for a closed Nambu-Goto string of tension T in phase-space form, with phase-space coordinates $X^m(\sigma)$ and $P_m(\sigma)$, where σ is the string coordinate. Show that $P + TX'$ and $P - TX'$ have zero Poisson bracket and hence that the algebra of Hamiltonian constraints is a direct sum of two isomorphic Lie algebras, with constraint functions that you should state. You need not compute the other Poisson brackets but you should state, without proof, the Poisson brackets of the Fourier modes of the constraint functions. Why does this tell you that the algebra is $\mathrm{Diff}_1 \oplus \mathrm{Diff}_1$ (where Diff_k is the algebra of vector fields on a k-dimensional manifold)?

What is the principle that determines the allowed boundary conditions at the end of an open string. Use this principle, and the phase-space form of the action, to show that one allowed choice is to fix the string at one end (e.g. to the origin of space coordinates) but to allow the other end to move freely. Why must the free end move at the speed of light?

Assuming that $X^0 = t$, find a solution of the NG equations of motion that describes a straight string of proper length L with one end fixed and the other end moving at the speed of light.

2

A point particle of mass μ in a D-dimensional Minkowski spacetime with cartesian coordinates x^m has the action $I = \int dt \left\{\dot{x}^m p_m - \frac{1}{2}e(p^2 + \mu^2)\right\}$. Write this action out using light-cone Minkowski coordinates $\{x^\pm, \mathbf{x}\}$. What is the canonical Hamiltonian in the light-cone gauge $x^+ = t$? Write down the time-dependent Schroedinger equation for this Hamiltonian and use it to show (for $\hbar = 1$) that the particle's wavefunction $\Psi(x)$ satisfies the Klein-Gordon equation $(\Box_D - \mu^2)\Psi = 0$.

An antisymmetric tensor field A, in a D-dimensional Minkowski spacetime, satisfies the equation

$$\partial^m F_{mnp} = \mu^2 A_{np}\,, \qquad F_{mnp} \equiv 3\partial_{[m}A_{np]}\,. \qquad (*)$$

Assuming that ∂_- is invertible, find the independent light-cone components of A and show that they satisfy the Klein-Gordon equation with mass μ. How many independent polarisation states are there? [*You may find it useful to consider first the massive vector equation* $\partial^m F_{mn} = \mu^2 A_n$]

In light-cone gauge (for oscillator modes) the phase-space action for the closed Nambu-Goto string, in D-dimensional Minkowski spacetime, takes the form

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k}\left(\dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} + \dot{\tilde{\boldsymbol{\alpha}}}_k \cdot \tilde{\boldsymbol{\alpha}}_{-k}\right) - \lambda_0 L_0 - \tilde{\lambda}_0 \tilde{L}_0 \right\} \qquad (\dagger)$$

where

$$L_0 = \frac{p^2}{8\pi T} + N\,, \qquad \tilde{L}_0 = \frac{p^2}{8\pi T} + \tilde{N}\,.$$

What is the physical interpretation of the various canonical variables of this action? Write down expressions for the "level numbers" N and $\tilde{N}$ in terms of them.

Now write down the canonical commutation relations that follow from the action ($\dagger$), and define the string oscillator vacuum. Assuming that the operators that replace N and $\tilde{N}$ annihilate the oscillator vacuum, prove that their eigenvalues are non-negative integers. What are the "level-one" states of the closed string? Explain why they are massless, and hence why the string ground state is a tachyon. What is the mass of a state at level N? Are there any states at level 2 that could be described by the equation $(*)$?

3

A point particle has action $I[x,p;e] = \int dt \left\{ \dot{x}^m p_m - \frac{1}{2} e(p^2 + m^2) \right\}$. Write down the (canonical) gauge transformations that leave invariant this action. Given that $e = s$ for some constant s, explain why s is gauge invariant. By considering $e(t)$ as a gauge transform of $e = s$, show that gauge-fixing in a path integral requires an additional Faddeev-Popov determinant in the path-integral measure. Explain (without proof) how this determinant can be exchanged for a contribution to the action involving Fadeev-Popov ghosts (b, c).

Now write down, in terms of Fourier modes of canonical variables, the phase-space action for the open Nambu-Goto string with free-end boundary conditions, in D spacetime dimensions. Write down the canonical commutation relations and define the oscillator vacuum $|0\rangle$. Show that $L_n|0\rangle = 0$ for $n > 0$, where L_n are the quantum operators obtained from the Fourier modes of the classical constraint functions. Why is the operator L_0 ambiguous? Write down, without proof, the commutation relations of the Virasoro algebra obeyed by the operators L_n, taking care to specify how you have defined L_0.

The BRST charge of the open string can be written in terms of the Fourier modes of the (anti)ghost fields (b, c) as

$$Q_{BRST} = \left(L_0 + \frac{1}{2} L_0^{(gh)} - 1 \right) c_0 + \sum_{m=1}^{\infty} \left[\left(L_{-m} + \frac{1}{2} L_{-m}^{(gh)} \right) c_m + c_{-m} \left(L_m + \frac{1}{2} L_m^{(gh)} \right) \right] ,$$

where

$$L_0^{(gh)} = \sum_{k=1}^{\infty} k \left(b_{-k} c_k + c_{-k} b_k \right) , \qquad L_m^{(gh)} = \sum_n (m - n)\, b_{m+n} c_{-n} \, , (m \neq 0) \, .$$

Write down the anticommutation relations of the (anti)ghost modes and define the (anti)ghost oscillator vacuum $|0\rangle_{gh}$. Let $|\Psi\rangle$ be a state in the Fock space acted on by the operators L_n. Find the conditions for the state $|\Psi\rangle \otimes |0\rangle_{gh}$ to be annihilated by Q_{BRST}. Why does your result imply that the state $|0\rangle \otimes |0\rangle_{gh}$ is a tachyon?

Given that

$$\left[L_m^{(gh)}, L_n^{(gh)} \right] = (m - n) \left(L_{m+n}^{(gh)} - \delta_{m+n} \right) - \frac{13}{6} \left(m^3 - m \right) \delta_{m+n} \, ,$$

find the algebra obeyed by the operators $\mathcal{L}_m = L_m + L_m^{(gh)}$. Stating without proof the implications for the algebra of the operators $\mathcal{L}_m$ implied by the condition $Q_{BRST}^2 = 0$, use your result to find the "critical" value of the spacetime dimension D.

4

Write an essay on amplitudes in closed string theory, and the structure of closed string perturbation theory, using the path-integral approach. Your essay should cover the following topics:

- You should explain how the scattering amplitude $A(p_1, \ldots, p_N)$ for N strings of momenta $(p_1, \ldots, p_N)$ in their tachyonic ground states can be represented by a Euclidean, conformal gauge, path integral. You should explain, in qualitative terms, why this leads to a formula for $A(p_1, \ldots, p_N)$ as a multiple integral over points on the Riemann sphere. [You need not give details of these integrals or of the resulting Virasoro–Shapiro formula.]

- You should discuss the Virasoro amplitude

$$A(s,t) = \frac{\Gamma(-1-t)\Gamma(-1-s)\Gamma(3+s+t)}{\Gamma(-2-s-t)\Gamma(s+2)\Gamma(t+2)} \,,$$

  for the scattering of four closed-string tachyons, with momenta (p_1, p_2, p_3, p_4) such that $p_1 + p_2 + p_3 + p_4 = 0$, where ($T$ is the string tension)

$$s = -\frac{1}{8\pi T}(p_1 + p_2)^2 \,, \qquad t = -\frac{1}{8\pi T}(p_1 + p_3)^2 \,.$$

  In particular, you should explain the physical significance of the invariants s and t, and of the poles in s at fixed t, and in t at fixed s.

- You should explain (focusing on the fields associated to the massless particles in the string spectrum) how amplitudes are encoded in the vacuum to vacuum amplitude of a string in background fields. You should explain, in qualitative terms, why this leads to an expansion in powers of α' (the Regge slope parameter) of an effective *spacetime* action S_{eff} for the massless fields.

- Finally, by considering how the dilaton couples to the string, you should explain how the path integral sum over Riemann surfaces of arbitrary genus g leads to a string-loop expansion of S_{eff} in powers of the string coupling constant g_s, which you should define.

END OF PAPER