UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 6 June, 2014 1:30 pm to 4:30 pm

PAPER 45

THE STANDARD MODEL

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the 4×4 matrix B to be one which transforms the Dirac matrices as follows

$$B\gamma^{\mu *}B^{-1} = \begin{cases} \gamma^\mu & \mu = 0 \\ -\gamma^\mu & \mu \neq 0 \end{cases} .$$

Given $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, show that $B\gamma^{5*}B^{-1} = \gamma^5$.

Consider a Dirac field ψ which can be written as an integral over plane wave solutions

$$\psi(x) \;=\; \sum_{p,s} \left[b^s(p)u^s(p)e^{-ip\cdot x} + d^{s\dagger}(p)v^s(p)e^{ip\cdot x} \right]$$

where $\sum_p \equiv \int \frac{d^3p}{(2\pi)^3(2p^0)}$. Show that the time-reversal transformation maps ψ and $\bar{\psi}$ as follows:

$$\hat{T}\psi(x)\hat{T}^{-1} = B\psi(x_T)$$
$$\hat{T}\bar{\psi}(x)\hat{T}^{-1} = \bar{\psi}(x_T)B^{-1}$$

where $p_T = (p^0, -\vec{p})$ and $x_T = (-x^0, \vec{x})$. (For simplicity let us assume intrinsic phases $\eta_T = 1$ throughout this problem.) You may use without proof

$$(-1)^{\frac{1}{2}-s}u^{-s*}(p_T) = -\gamma^5 C u^s(p)$$
$$(-1)^{\frac{1}{2}-s}v^{-s*}(p_T) = -\gamma^5 C v^s(p) .$$

You should have obtained a relation between B and C. Use this and the defining property of B stated at the start of this problem to show that

$$C\gamma^{\mu T}C^{-1} = -\gamma^\mu$$

where $\gamma^{\mu T}$ is the transpose of γ^μ.

Consider the decay of a neutron n to a proton p: $n \to pe^-\bar{\nu}_e$, treating the neutron and proton (as well as the electron and anti-neutrino) as Dirac fields. Given that the effective interaction mediating this decay is

$$\mathcal{L}_I \;=\; -\frac{G_F}{\sqrt{2}} J^\dagger_{\text{lept},\mu} J^\mu_{\text{pn}} \;+\; \text{h.c.},$$

with $J^\mu_{\text{lept}} \;=\; \bar{\nu}_e\gamma^\mu(1-\gamma^5)e$ and $J^\mu_{\text{pn}} \;=\; \bar{p}\gamma^\mu(g_V + g_A\gamma^5)n$, derive a condition which g_A/g_V must satisfy if this interaction is to be invariant under time-reversal.

2

The electroweak theory of the Standard Model consists of an $SU(2)_L \times U(1)_Y$ gauge theory which undergoes spontaneous symmetry breaking via the Higgs mechanism. Let W_μ^a $(a = 1, 2, 3)$ be the $SU(2)_L$ gauge bosons with coupling g and B_μ be the $U(1)_Y$ gauge boson with coupling g'. The scalar field ϕ transforms as a doublet under $SU(2)_L$, has hypercharge $\frac{1}{2}$, and has a Lagrangian of the form

$$\mathcal{L}_\phi = (D_\mu \phi)^\dagger (D^\mu \phi) - \mu^2 |\phi|^2 - \lambda |\phi|^4 \quad (\lambda > 0)\,.$$

Explicitly write down the terms arising from covariant differentiation of ϕ, i.e. from $D_\mu \phi$.

Show that mass terms for 3 out of 4 gauge bosons are generated if the scalar field acquires a nonzero vacuum expectation value. Be clear about the relation between the original fields (W_μ^a, B_μ) and the ones after symmetry breaking, $(W_\mu^\pm, Z_\mu^0, A_\mu)$.

Show how the electroweak theory includes the electron e and electron neutrino ν_e. In particular, write down gauge-invariant terms in the electroweak Lagrangian which contain the coupling of these fermions to the gauge bosons and others which contain the fermion-scalar interactions. How does the latter lead to a nonzero electron mass?

3

Consider the weak semileptonic decay

$$\bar{K}^0 \to \pi^+ e^- \bar{\nu}_e$$

where $\bar{K}^0$ and π^+ are pseudoscalar mesons. Assume this decay proceeds due to the interaction

$$\mathcal{L}_W^{\text{eff}} = -\frac{G_F}{\sqrt{2}} J^\dagger_{\text{lept},\mu} J^\mu_{\text{had}} + \text{h.c.}$$

with $J^\mu_{\text{lept}} = \bar{\nu}_e \gamma^\mu (1-\gamma^5) e$ and $J^\mu_{\text{had}} = V_{us} \bar{u} \gamma^\mu (1-\gamma^5) s$. [Ignore $K^0 - \bar{K}^0$ mixing.]

Explain why the following 2 equalities hold

$$\begin{aligned} \langle \pi^+(k) | \bar{u}\gamma^\mu (1-\gamma^5) s | \bar{K}^0(p) \rangle &= \langle \pi^+(k) | \bar{u}\gamma^\mu s | \bar{K}^0(p) \rangle \\ &= (p+k)^\mu f_+(q^2) + (p-k)^\mu f_-(q^2) \end{aligned}$$

where $q \equiv p - k$ and $f_+(q^2)$ and $f_-(q^2)$ are scalar (dimensionless) functions of q^2.

Treating the electron and anti-neutrino as massless, derive the invariant scattering amplitude $\mathcal{M}$. In preparation to calculate the decay rate $\Gamma(\bar{K}^0 \to \pi^+ e^- \bar{\nu}_e)$, find and justify an expression for $\mathcal{M}$ which is proportional to $f_+(q^2)$ (i.e. has no $f_-(q^2)$ term).

Show that the decay rate $\Gamma(\bar{K}^0 \to \pi^+ e^- \bar{\nu}_e)$ for this process can be written as an integral over pion momentum

$$A \int \frac{d^3 k}{k^0} \left[(p \cdot q)^2 - q^2 m_K^2 \right] |f_+(q^2)|^2$$

where A is a dimensionful constant which you should determine.

By working in the $\bar{K}^0$ rest frame, or otherwise, show that the decay rate can be expressed as

$$\Gamma = B \int_a^b dq^2 \lambda^{3/2} |f_+(q^2)|^2$$

where the kinematic variable λ is defined as

$$\lambda = (q^2)^2 + m_K^4 + m_\pi^2 - 2q^2 m_K^2 - 2q^2 m_\pi^2 - 2m_K^2 m_\pi^2 \,.$$

You should determine the dimensionful constant B and the limits of integration a and b.

[You may use without proof the following:

$$\Gamma = \frac{1}{2m_K} \int \frac{d^3 k}{(2\pi)^3 2k^0} \frac{d^3 q_1}{(2\pi)^3 2q_1^0} \frac{d^3 q_2}{(2\pi)^3 2q_2^0} (2\pi)^4 \delta^{(4)}(p - k - q_1 - q_2) \sum_{\text{spins}} |\mathcal{M}|^2$$

$$\begin{aligned} \operatorname{Tr} \gamma^\alpha \gamma^\beta \gamma^\tau \gamma^\delta &= 4(g^{\alpha\beta} g^{\tau\delta} - g^{\alpha\tau} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\tau}) \\ \operatorname{Tr} \gamma^\alpha \gamma^\beta \gamma^\tau \gamma^\delta \gamma^5 &= 4i\epsilon^{\alpha\beta\tau\delta} \end{aligned}$$

$$\begin{aligned} \int \frac{d^3 q_1}{|\vec{q}_1|} \frac{d^3 q_2}{|\vec{q}_2|} \delta^{(4)}(Q - q_1 - q_2)\, q_{1\mu} q_{2\nu} &= \frac{\pi}{3} Q_\mu Q_\nu + \frac{\pi}{6} g_{\mu\nu} Q^2 \\ \int \frac{d^3 q_1}{|\vec{q}_1|} \frac{d^3 q_2}{|\vec{q}_2|} \delta^{(4)}(Q - q_1 - q_2) &= 2\pi \quad] \end{aligned}$$

4

Renormalized coupling constants $g_i(\mu)$ $(i = 1, 2, 3)$ generally depend on a renormalization scale μ. For small values of the couplings, the scale dependence can be written to one-loop order as

$$\mu \frac{d}{d\mu} g_i(\mu) \;=\; b_i g_i^3 + O(g_i^5)\,.$$

What are the consequences of b_i being positive vs. negative?

For $\alpha_i = g_i^2/4\pi$, derive an expression relating $\alpha_i(m_Z)$ to $\alpha_i(\mu)$. (m_Z is the mass of the Z boson.)

Let α_1, α_2, and α_3 be the coupling constants of the respective Standard Model gauge groups $U(1)_Y$, $SU(2)_L$, and $SU(3)_c$. Suppose there exists a scale $M_{GUT} > m_Z$ at which the following holds:

$$\frac{5}{3}\alpha_1(M_{GUT}) \;=\; \alpha_2(M_{GUT}) \;=\; \alpha_3(M_{GUT})\,.$$

Show that this implies

$$\alpha_3^{-1}(m_Z) \;=\; \alpha_2^{-1}(m_Z) + \frac{b_3 - b_2}{\frac{3}{5}b_1 - b_2}\left[\frac{3}{5}\alpha_1^{-1}(m_Z) \;-\; \alpha_2^{-1}(m_Z)\right].$$

Now let us work to another order, considering a single coupling α. Define $a = \alpha/4\pi$. Given

$$\mu \frac{d}{d\mu} a \;=\; -\beta_0 a^2 - \beta_1 a^3\,,$$

show that

$$a^{-1}(\mu) = \beta_0 \log\frac{\mu}{\Lambda} \;+\; \frac{\beta_1}{\beta_0}\log\log\frac{\mu}{\Lambda} \;+\; O\left(1/\log\frac{\mu}{\Lambda}\right)$$

for a suitable choice of Λ.

END OF PAPER