UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 9:00 am to 12:00 pm

PAPER 44

ADVANCED QUANTUM FIELD THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let M be an $N \times N$ Hermitian matrix and let

$$V(\mathrm{M}) = \frac{1}{2}\mathrm{tr}(\mathrm{M}^2) + \frac{g}{4}\mathrm{tr}(\mathrm{M}^4)$$

where g is a constant.

a) Show that $V(\mathrm{M})$ is invariant under $\mathrm{M} \to U^\dagger \mathrm{M} U$ where U is a unitary matrix, and hence that $V(\mathrm{M})$ depends only on the eigenvalues $\{\lambda_i\}$ of M.

b) Viewing $V(\mathrm{M})$ as the action for a zero dimensional QFT, obtain an expression for the propagator $\langle \mathrm{M}^i{}_j\, \mathrm{M}^k{}_l \rangle$ correct to lowest non-trivial order in g. [*Consider only connected diagrams.*]

c) Now let B, C and H be further $N \times N$ matrices, each of which have zeros all along the leading diagonal. Let the elements of B and C be fermionic variables, while the entries of H are bosonic. Consider the matrix integral

$$Z(g) = \int \frac{[\mathrm{dM}\,\mathrm{dB}\,\mathrm{dC}\,\mathrm{dH}]}{(2\pi \mathrm{i})^{N(N-1)}} \exp\Big(- N V(\mathrm{M}) + \mathrm{i}\,\mathrm{tr}(\mathrm{HM}) + \mathrm{tr}(\mathrm{B}[\mathrm{M},\mathrm{C}])\Big)$$

where the measure $[\mathrm{dM}\,\mathrm{dB}\,\mathrm{dC}\,\mathrm{dH}]$ indicates an integral over each entry of M, and the off-diagonal entries of B, C and H. Obtain the effective action for the eigenvalues $\{\lambda_i\}$ of M. [*Do not attempt to perform the path integral over these eigenvalues.*]

2

a) Briefly explain the main idea of Wilson's picture of renormalization. What does it mean for an coupling to be *relevant*, *irrelevant* or *marginal*?

b) A Euclidean scalar field theory has action $S = S_0 + S_1$, where $S_1(\phi, \Lambda)$ is a sum of polynomial interactions for ϕ and where the kinetic part of the action is

$$S_0(\phi, \Lambda) = \frac{1}{2}\int \mathrm{d}^4x\ \phi(x) f\left(\frac{-\Box + m^2}{\Lambda^2}\right)\phi(x)$$

with $f(z)$ any smooth function that obeys $f(z) = \Lambda^2 z$ for $z \leqslant 1$ and blows up rapidly when $z > 1$. Λ is a cutoff. Neglecting interactions, what is the momentum space propagator for this theory? Show that field modes whose energies are high compared to Λ do not propagate.

c) Show that if the interaction terms obey

$$\Lambda\frac{\partial}{\partial\Lambda}\mathrm{e}^{-S_1} = -\frac{1}{2}\int \mathrm{d}^4p\,(2\pi)^4\,\Lambda\frac{\partial(1/f)}{\partial\Lambda}\left[\frac{\delta^2\,\mathrm{e}^{-S_1}}{\delta\tilde{\phi}(p)\delta\tilde{\phi}(-p)}\right] \qquad (\dagger)$$

then the partition function $Z(\Lambda)$ is independent of the cutoff. [*Neglect a contribution which simply rescales* Z.]

d) By carrying out the field derivatives, interpret equation (†) in terms of Feynman diagrams.

3

a) Explain how the one-particle-irreducible electron self-energy $\Sigma(\not{p})$ is related to the exact momentum space electron propagator

$$\mathrm{i}G(\not{p}) = \int \mathrm{d}^D x \, \mathrm{e}^{\mathrm{i}p\cdot(x-y)} \, \langle \psi(x)\bar{\psi}(y)\rangle$$

in QED. What is the relation between this propagator and the physical mass of the electron?

b) Briefly explain the difference between the *on-shell* and *minimal subtraction* (or $\overline{\mathrm{MS}}$) renormalization schemes.

c) In $D = 4 - \epsilon$ dimensions, the 1-loop contribution to the electron self-energy in QED is

$$\Sigma^{1\,\mathrm{loop}}(\not{p}) = \frac{e^2}{8\pi^2}\int_0^1 \mathrm{d}x \; (x\not{p} - 2m)\left[\frac{2}{\epsilon} + \ln\frac{4\pi\mathrm{e}^{-\gamma_\mathrm{E}}\mu^2}{(1-x)(m^2 - p^2 x)}\right] ,$$

where e is a dimensionless coupling and m is the renormalized electron mass. μ is an arbitrary renormalization scale while $\gamma_\mathrm{E} \approx 0.577$ is the Euler-Mascheroni constant. Which counterterms are needed to absorb the divergent part of this integral?

d) Show that in the $\overline{\mathrm{MS}}$ renormalization scheme, the renormalized electron mass m is related to the physical electron mass m_{phys} by

$$m = m_{\mathrm{phys}}\left[1 - \frac{e^2}{16\pi^2}\left(5 + 3\ln\frac{\mu^2}{m^2_{\mathrm{phys}}}\right)\right]$$

up to terms of order e^4.

4

a) Consider an $SU(N)$ gauge theory. Define the covariant derivative $D_\mu\psi$ acting on a field ψ in the adjoint representation of the gauge group.

b) BRST transformations are defined as

$$\delta A_\mu = \epsilon\, D_\mu c\ , \qquad \delta c = \frac{\mathrm{i}}{2}\epsilon\,[c, c]$$
$$\delta\bar{c} = \epsilon\, h\ , \qquad \delta h = 0$$

where c and $\bar{c}$ are ghost and antighost fields while h is an auxiliary (Nakanishi–Lautrup) field. ϵ is an anticommuting parameter. Show that these transformations are nilpotent when acting on an arbitrary operator $\mathcal{O}(A_\mu, c, \bar{c}, h)$ that is polynomial in the fields.

c) Obtain the full action, including ghosts, for a pure (with no matter) Abelian gauge theory where the gauge field is constrained to obey $\partial^\mu A_\mu + A^\mu A_\mu = 0$. What are the ghost–antighost propagator and the ghost–antighost–gauge field vertex in momentum space?

d) Explain why correlation functions of gauge invariant operators are independent of this gauge fixing condition.

END OF PAPER