UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 2 June, 2014 1:30 pm to 3:30 pm

PAPER 42

STATISTICAL FIELD THEORY

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Give an account of the Landau–Ginsburg (LG) theory of phase transitions which should include a discussion of the following points:

(i) The idea of an order parameter;

(ii) The distinction between first-order and continuous phase transitions and how their occurrence is predicted in LG theory;

(iii) the reason why a line of first-order phase transitions must terminate in a critical point associated with a continuous phase transition;

(iv) The idea of *critical exponents* and how they may be derived;

(v) The features of a tricritical point, how it occurs in LG theory and an explanation of the features of the 3D phase diagram containing a tricritical point.

State what is meant by the *scaling hypothesis*. For a system described by a single scalar field, show that LG theory predicts that in the neighbourhood of an ordinary continuous phase transition the equilibrium free energy, A, can be written as

$$A \;=\; a|t|^2 f_{\lessgtr}\left(b\frac{h}{|t|^{3/2}}\right) , \qquad (*)$$

where a and b are real positive constants, t is the reduced temperature and h is the magnetic field, i.e. the external field conjugate to the order parameter. Explain the meaning of the subscript $\lessgtr$ on $f_{\lessgtr}$.

Use the expression $(*)$ to compute two critical exponents of your choice.

How should $(*)$ be modified to accommodate anomalous scaling behaviour? In this case show that the scaling hypothesis predicts that the scaling relations

$$\alpha + 2\beta + \gamma = 2 \;, \qquad \beta\delta = \beta + \gamma \;,$$

hold where the critical exponents $\alpha, \beta, \gamma, \delta$ should be defined.

2

A statistical field theory in D dimensions is defined on a cubic lattice of spacing a with N sites and with field variable $\sigma_{\mathbf{n}}$, which may be continuous, on the $\mathbf{n}$-th site. The Hamiltonian density is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$\mathcal{H}(\mathbf{u}, \sigma) \;=\; \sum_i \, u_i O_i(\{\sigma\}) \;,$$

where the u_i are coupling constants with $\mathbf{u} = (u_1, u_2, \ldots)$. In particular, $\mathcal{H}$ contains the term $-h \sum_{\mathbf{n}} \sigma_{\mathbf{n}}$ where h is the magnetic field. The partition function is given by

$$\mathcal{Z}(\mathbf{u}, C, N) \;=\; \sum_\sigma \, \exp(-\beta\mathcal{H}(\mathbf{u}, \sigma) - \beta N C) \;.$$

Define the two-point correlation function $G(\mathbf{r})$ for the theory and state how the correlation length ξ parametrizes its behaviour as $|\mathbf{r}| \to \infty$. State how the susceptibility χ can be expressed in terms of $G(\mathbf{r})$.

Explain how the renormalization group (RG) transformation may be defined in terms of a blocking kernel which, after p iterations, yields a blocked partition function $\mathcal{Z}(\mathbf{u}_p, C_p, N_p)$ which predicts the same large-scale properties for the system as does $\mathcal{Z}(\mathbf{u}, C, N)$. State how a and N rescale in terms of the RG scale factor b.

Derive the RG equation for the free energy $F(\mathbf{u}_p, C_p)$, and explain how it may be expressed in terms of a singular part, $f(\mathbf{u})$, which obeys the RG equation

$$f(\mathbf{u}_0) \;=\; b^{-pD} f(\mathbf{u}_p) \;+\; \sum_{j=0}^{p-1} b^{-jD}\, g(\mathbf{u}_j) \;, \qquad p > 0 \;.$$

What is the origin of the function $g(\mathbf{u})$ which determines the inhomogeneous part of the transformation?

In the context of the RG equations, explain the ideas of a *fixed point*, *relevant* and *irrelevant* operators, a *critical surface* and a *repulsive trajectory*. Sketch some typical RG flows near to a critical surface.

Show how the critical exponents characterizing a continuous phase transition may be derived. In the case where there are two relevant couplings $t = (T - T_C)/T_C$ and h, derive the scaling hypothesis for the singular part, F_s, of the free energy:

$$F_s \;=\; |t|^{D/\lambda_t} f_{\pm}\left(\frac{h}{|t|^{\lambda_h/\lambda_t}}\right) \;,$$

where the meanings of λ_t, λ_h should be explained.

The following critical exponents ν, γ, α are defined for $h = 0$:

$$\xi \sim |t|^{-\nu}, \quad \chi \sim |t|^{-\gamma}, \quad C_V \sim |t|^{-\alpha},$$

where χ is the susceptibility and C_V is the specific heat at constant volume. Establish the scaling relation $\alpha = 2 - D\nu$.

According to the scaling hypothesis the correlation function for $|\mathbf{r}| \ll \xi$ takes the form

$$G(|\mathbf{r}|) = \frac{1}{|\mathbf{r}|^{D-2+\eta}} f_G(|\mathbf{r}|/\xi).$$

What form is $G(|\mathbf{r}|)$ expected to take when $|\mathbf{r}| \gg \xi$? From this parametrization obtain an expression for the susceptibility and derive the scaling relation $\gamma = (2-\eta)\nu$.

In the case that $\sigma_{\mathbf{n}}$ is a continuous field variable explain briefly how the exponent η is related to the scaling renormalization of the field.

3

A statistical system in D dimensions and at temperature T, is described by a scalar field theory whose effective Hamiltonian is defined by

$$\begin{aligned} H(\phi) &= \int_{\Lambda^{-1}} d^D x \, \mathcal{H}(\Lambda, \phi(\mathbf{x})) \, , \\ \mathcal{H}(\Lambda, \phi(\mathbf{x})) &= \frac{1}{2}\alpha^{-1}(\Lambda, T)(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2}m^2(\Lambda, T)\phi^2(\mathbf{x}) + \frac{1}{4!}g(\Lambda, T)\phi^4(\mathbf{x}) + \dots \, , \end{aligned}$$

where Λ is the ultra-violet cut-off and $\mathcal{H}$ is the Hamiltonian density. The partition function is

$$\mathcal{Z} = \int \{d\phi\} e^{-H(\phi)} \, .$$

Why do the coupling constants depend on Λ?

By giving an example of a blocking transformation explain how a Renormalization Group (RG) strategy for successively integrating out high-momentum modes may be applied to this model.

By making suitable assumptions show how the Landau-Ginsburg theory of phase transitions may be derived using the RG in the context of this model.

In the case of a ϕ^4 scalar field theory, the two-point function $G(\mathbf{x})$ and its Fourier transform $\tilde{G}(\mathbf{p})$ are defined by

$$G(\mathbf{x}) = \langle \phi(0) \, \phi(\mathbf{x}) \rangle_c \, , \qquad \tilde{G}(\mathbf{p}) = \int d^D x \, e^{-i\mathbf{p}\cdot\mathbf{x}} G(\mathbf{x}) \, .$$

State what is meant by the truncated two-point function $\tilde{\Gamma}(\mathbf{p})$.

Using perturbation theory explain how $\tilde{\Gamma}(\mathbf{p})$ may be written as

$$\tilde{\Gamma}(\mathbf{p}) = \tilde{G}_0^{-1}(\mathbf{p}) + \delta m^2 + \Sigma(\mathbf{p}) \, ,$$

where the meaning of each of the terms in this expression should be clearly given. You may quote the rules of perturbation theory without derivation.

Hence show to one-loop order that

$$m^2(0, T) = m^2(\Lambda, T) + \frac{g}{2} \int^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2(0, T)} \, .$$

Show that this result is consistent with the Landau-Ginsburg assumption that $m^2(0, T) \sim (T - T_C)$ only for $D > D_C$, where the value of D_C for an ordinary critical point should be calculated.

Describe briefly how the value of D_C for a tricritical point is calculated and determine its value.

END OF PAPER