UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 3 June, 2014 9:00 am to 12:00 pm

PAPER 41

SYMMETRIES, FIELDS AND PARTICLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

"An abstract Lie algebra L is a vector space with a bracket $[\ ,\ ] : L \times L \to L$, linear in both entries, satisfying antisymmetry and the Jacobi identity." Explain what is meant by antisymmetry and the Jacobi identity.

A vector space $\widetilde{L}$ of complex square matrices has bracket defined by

$$[A, B] = AB - BA\,.$$

Explain what requirements need to be checked to ensure that $\widetilde{L}$ is a Lie algebra. Find $L\big(SU(n)\big)$, the Lie algebra of the matrix Lie group $SU(n)$, and show that it satisfies these requirements.

Consider the vector space $\mathbb{R}^3$, with $\mathbf{x}$ denoting a general element. Show that the bracket

$$[\mathbf{x}, \mathbf{y}] = \mathbf{x} \times \mathbf{y}$$

(with $\mathbf{x} \times \mathbf{y}$ the usual cross product) makes $\mathbb{R}^3$ into a Lie algebra. How does this definition relate to the Lie algebra $L\big(SU(2)\big)$?

2

Pure electromagnetic theory has a gauge potential a_μ, field tensor $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, and Lagrangian density

$$\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu}\,.$$

Explain how to include a charged scalar field ϕ in the theory, coupled to the gauge potential in a gauge invariant way, and verify the gauge invariance. Give examples of such theories that (a) *do*, and (b) *do not*, exhibit the Higgs mechanism.

Discuss the properties of the particles that occur in the pure electromagnetic theory, and in the theories which have a coupled scalar field and gauge potential.

Write down a physically sensible example of a Lagrangian density for a gauge invariant theory with two charged scalar fields ϕ and ψ coupled to the gauge potential a_μ. You should assume that ψ has double the charge of ϕ, so that under a gauge transformation, $\phi(x) \to e^{i\alpha(x)}\phi(x)$ and $\psi(x) \to e^{2i\alpha(x)}\psi(x)$. You should include in your Lagrangian density one or more non-trivial terms directly coupling ϕ to ψ.

3

The quarks u, d and s combine into a triplet $\underline{3}$ of the approximate, flavour $SU(3)$ symmetry. Baryons are composed of three quarks. Explain how $SU(3)$ baryon multiplets $\underline{10}$, $\underline{8}$ and $\underline{1}$ arise, and use the $SU(3)$ weight diagrams to give one or two examples of the quark content of baryons in each of these multiplets.

State the Pauli principle in the context of 3-quark baryon states. Discuss briefly the consequences of the Pauli principle for baryons in each of the $SU(3)$ multiplets $\underline{10}$, $\underline{8}$ and $\underline{1}$.

It has been proposed that there might be pentaquark baryons, composed of four quarks and an antiquark. Show that some pentaquark baryons would lie in a $\overline{\underline{10}}$ multiplet of flavour $SU(3)$. How many of these ten pentaquark baryons have Y and I_3 quantum numbers distinct from any 3-quark baryon? Would these be long- or short-lived?

4

The Poincaré Lie algebra has basis elements P^τ and $M^{\rho\sigma}$ (with $M^{\rho\sigma} = -M^{\sigma\rho}$) and brackets

$$\begin{aligned}
[P^\tau, P^\mu] &= 0\,, \\
[M^{\rho\sigma}, P^\tau] &= \eta^{\sigma\tau} P^\rho - \eta^{\rho\tau} P^\sigma\,, \\
[M^{\rho\sigma}, M^{\tau\mu}] &= \eta^{\sigma\tau} M^{\rho\mu} - \eta^{\rho\tau} M^{\sigma\mu} + \eta^{\rho\mu} M^{\sigma\tau} - \eta^{\sigma\mu} M^{\rho\tau}\,,
\end{aligned}$$

where $\eta^{\sigma\tau} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric. Among these basis elements identify the rotation generators J_1, J_2, J_3 and verify that they satisfy

$$[J_1, J_2] = J_3\,.$$

The universal enveloping algebra contains operators $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\tau} M^{\nu\rho} P^\tau$. Show that $W_\mu P^\mu = 0$ and that

$$[W_\mu, P^\sigma] = 0\,.$$

A particle of mass $m > 0$ is at rest in a spin state $|\psi\rangle = |j, j_3\rangle$. Show that $|\psi\rangle$ is an eigenstate of W_0 and W_3, and find the eigenvalues.

A massless particle has momentum $p^\mu = (k, 0, 0, k)$ and is in a state $|\psi\rangle$ with helicity j_3. Explain what is meant by helicity. Show that $|\psi\rangle$ is an eigenstate of W_0 and W_3, and find the eigenvalues.

Show that your eigenvalues are consistent with $W_\mu P^\mu = 0$.

END OF PAPER