UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 29 May, 2014 9:00 am to 12:00 pm

PAPER 40

QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

In this paper, the sign convention for the signature of spacetime as $(-+++)$ has been used and the anti-commutator of a pair of Dirac gamma matrices taken to be $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$.

However any other consistent set of sign conventions may be used as long as they are clearly stated.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The Lagrangian for a massive vector field A_a, with field strength $F_{ab} = \partial_a A_b - \partial_b A_a$, is given by

$$\mathcal{L} = -\frac{1}{4} F^{ab} F_{ab} - \frac{1}{2} m^2 A^a A_a.$$

(i) Derive the equations of motion for the field A_a.

(ii) Show that $\partial_a A^a = 0$.

(iii) Show that this implies that $q_a \widehat{A}^a = 0$ where the hat on A indicates that the Fourier transform of A has been taken and that it carries momentum q.

(iv) Find a projection operator in momentum space that when acting on a vector projects out the part proportional to $q_a \widehat{A}^a$.

(v) The vector field has three polarization states. Assuming that the spatial part of the momentum is $\mathbf{q}$ along the z-axis, find the three polarization vectors.

(vi) Find an expression for

$$\Delta(x, y) = \langle 0 | T\, A_a(x) A_b(y) | 0 \rangle.$$

in terms of its Fourier transform.

2

The Lagrangian for a scalar field ϕ interacting with a spin-$\frac{1}{2}$ field ψ is

$$\mathcal{L} = -\frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{2} m^2 \phi^2 + \bar{\psi}(\gamma^a \partial_a + M)\psi + g\bar{\psi}\gamma^5 \psi \phi.$$

(i) What are the complete Feynman rules for this theory.

(ii) A fermionic particle has 4-momentum p^a It collides with a scalar of momentum k^a and the final state is a fernionic particle of 4-momentum q^a and a scalar with 4-momentum l^a. The spin of the initial state fermion is s and the final state fermion is s'.

Find an expression for the scattering amplitudes at tree level.

(iii) Explain how to find the spin-averaged total cross section for this process given that the transition amplitude for the process is T. You are not required to evaluate the total cross-section.

(iv) Suppose that one changed the final state fermion to an antiparticle. What difference to your calculation would this make?

3

(i) Describe the process of integration for Grassmann variables.

(ii) Prove that

$$\int d\eta \int d\bar{\eta}\ e^{\bar{\eta}A\eta} = \det A$$

where η and $\bar{\eta}$ are a set of N Grassmann numbers and A is a diagonalizable $N \times N$ matrix of ordinary numbers.

(iii) The Lagrangian for a free spin-$\frac{1}{2}$ particle is

$$\mathcal{L} = \bar{\psi}(\gamma^a \partial_a + M)\psi.$$

The path integral for fermions is given by

$$Z(J, \bar{J}) = \int D\psi\ D\bar{\psi}\ e^{i\int(\mathcal{L} + \bar{\psi}J(x) + \bar{J}(x)\psi)}$$

where J and $\bar{J}$ are sources for $\bar{\psi}$ and ψ respectively. Starting from Z, evaluate the Feynman propagator for the fermions.

(iv) What formally is $Z(0,0)$? Compare this result for scalar fields and comment on its relevance to vacuum energy.

4

The Lagrangian for a free scalar field ϕ is

$$\mathcal{L} = -\frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2}m^2\phi^2.$$

Write an essay describing how to apply the method of canonical quantisation to this system. Your essay should include at least a discussion of the nature of particles, their behaviour in spacetime and Bose statistics.

END OF PAPER