UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 9:00 am to 11:00 am

PAPER 4

TOPICS IN INFINITE GROUPS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Given any two groups G_1 and G_2, describe the construction of the *free product* $G_1 * G_2$ and state the universal property of free products.

State and prove Klein's combination theorem for subgroups G_1, G_2 of homeomorphisms of a topological space X. Use it to give an example of a free product $G_1 * G_2$ where neither G_1 nor G_2 is the trivial group.

Define what it means for a group to be *finitely presented*. Given finite presentations for two groups G and H, state and prove a result that gives a presentation for $G * H$, and one for the semi-direct product $G \rtimes_\phi H$.

Show that the semidirect product $F_3 \rtimes_\phi C_3$, where F_3 is free on x, y, z, the cyclic group C_3 is generated by c and $\phi(c)$ sends x to y to z to x, is also a free product of two finitely presented, non trivial groups.

2

Define a *soluble* group and show that the property of solubility is preserved under subgroups, quotients and extensions.

Given a finite index subgroup H of a group G, show that:

(i) If S is any subgroup of G then $H \cap S$ has finite index in S.
(ii) If L also has finite index in G then so does $H \cap L$.
(iii) If θ is a surjective homomorphism from G to some other group R then $\theta(H)$ has finite index in R.

Now define a *virtually* soluble group and show that the property of being virtually soluble is also preserved under subgroups, quotients and extensions.

Give with brief justification an example of any group that is not virtually soluble but which does not contain a non abelian free subgroup.

3

Prove that if a subgroup H of a group G has finite index in G then there is a normal subgroup N of finite index in G which is contained in H.

Describe Higman's construction of an infinite, finitely presented group with no proper finite index subgroups. [You may assume standard properties of HNN extensions and free products with amalgamation.]

Although there exist infinite groups where all non-identity elements are conjugate, such a group must be torsion free: let G be an infinite group such that all non-identity elements are conjugate in G and suppose that $g \in G \setminus \{e\}$ has finite order n. What can we say about the order of any element in G?

Now assume that $n \geqslant 3$. By taking an element x that conjugates g to g^2 and considering $x^n g x^{-n}$, obtain a contradiction on the order of g.

Why do we need $n \neq 2$ in the above argument? Explain how we can eliminate the case $n = 2$ anyway.

4

Define what it means for a group G to be *residually finite* and to be *Hopfian.*

Prove that a finitely generated group has only finitely many subgroups of a given index n for each $n \in \mathbb{N}$ and thus that a finitely generated, residually finite group is Hopfian.

State and prove conditions under which a semidirect product is residually finite.

Which of the following groups are residually finite? (You may quote any results needed on HNN extensions.)

(i) The Baumslag–Solitar group $BS(1, -1)$ given by the presentation

$$BS(1,-1) = \langle a, b \mid bab^{-1} = a^{-1} \rangle.$$

(ii) The Baumslag–Solitar group $BS(3, 4)$ given by the presentation

$$BS(3,4) = \langle a, b \mid ba^3b^{-1} = a^4 \rangle.$$

(iii) The group generated by the matrices

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

5

Given a presentation consisting of finitely many generators (but maybe infinitely many relators) along with a prime p, define the *p-deficiency* of this presentation.

Show how this concept is used to prove the existence of an infinite, finitely generated group which is torsion (namely every element has finite order).

For each group given below, say whether there exists a presentation for the group and a prime p such that this presentation has p-deficiency at least 1.

(i) The group G_1 given by the presentation

$$\langle v, w, x, y, z \mid vwv^{-1} = w^2, wxw^{-1} = x^2, xyx^{-1} = y^2, yzy^{-1} = z^2, zvz^{-1} = v^2 \rangle.$$

(ii) The group G_2 given by the presentation

$$\langle a, t \mid tat^{-1} = a^{-1}, t^2 \rangle.$$

(iii) The group G_3 given by the presentation

$$\langle x, y \mid x^2, y^2, (xy)^{1024} \rangle.$$

(iv) The group G_4 given by the presentation

$$\langle x, y, z \mid x^2, y^3, z^{36}, (xy)^{48}, (yz)^{48}, (zx)^{72} \rangle.$$

END OF PAPER