UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 5 June, 2014 9:00 am to 11:00 am

PAPER 39

DESIGNING ONLINE CONTESTS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

*This is an **OPEN BOOK** examination.*
Candidates may only bring into the examination handwritten or personally typed lecture notes and handouts from this course.
No other material, or copies thereof, are allowed.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a contest among $n \geqslant 2$ players with valuations $v_1 \geqslant v_2 \geqslant \cdots \geqslant v_n > 0$ where each player incurs a unit marginal cost of production. A unit prize is allocated according to the following prize allocation function, for $\delta \geqslant 0$,

$$x_i(b_1, b_2, \dots, b_n) = \frac{b_i}{\sum_{j \in N} b_j + \delta}, \text{ for } i \in N.$$

Note that this corresponds to proportional allocation with one extra player who invests a constant effort in the amount of δ. This accommodates usual proportional allocation as a special case when $\delta = 0$.

Show that in pure-strategy Nash equilibrium for every $i \in N$, it either holds

$$\begin{aligned} b_i &= (R + \delta)\left(1 - \frac{R+\delta}{v_i}\right) \text{ and } b_i > 0 \\ \text{or that } v_i &\leqslant R + \delta \text{ and } b_i = 0 \end{aligned}$$

where $R = \sum_{j \in N} b_j$.

Let $\hat{n}$ be the number of active players (those who invest strictly positive effort in pure-strategy Nash equilibrium) and $\bar{v}_{\hat{n}} = \hat{n} / \sum_{i=1}^{\hat{n}} 1/v_i$.

Give an explicit characterization of the expected total effort R in terms of parameters $\hat{n}$, $\bar{v}_{\hat{n}}$ and δ. It is not needed to explain that active players are the players with $\hat{n}$ largest valuations.

Give an explicit characterization of $\hat{n}$ in terms of valuations $v_1, v_2, \dots, v_n$ and δ.

2

Consider a contest among n players, for $n \geqslant 2$, with private valuations that are assumed to be independent and identically distributed according to the uniform distribution on $[0, 1]$. Assume that players incur unit marginal costs of production and that the contest allocates $1 \leqslant m < n$ identical prizes to a set of players who invest largest efforts, each of value $w(m/n)$ for a given continuously differentiable function $w : \mathbb{R}_+ \to \mathbb{R}_+$.

Show that if $w'(x)x(1-x) + w(x)(1-2x) \leqslant 0$, for all $x \in [0, 1]$, then allocating a single prize is optimal with respect to the expected total effort in Bayes–Nash equilibrium.

Specifically, for $w(x) = 1/x^\alpha$ with $\alpha > 0$, show that allocating a single prize is optimal with respect to the expected total effort in Bayes-Nash equilibrium if $\alpha \geqslant 1$. On the other hand, if $\alpha < 1$, show that optimal value of m is either the largest integer m such that $m \leqslant [(1-\alpha)/(2-\alpha)]n$ or the smallest integer m such that $m > [(1-\alpha)/(2-\alpha)]n$.

3

Consider a contest between two players with private valuations v_1 and v_2 for player 1 and player 2, respectively, which are assumed to be independent and identically distributed according to distribution function F. Assume that F is increasing, continuously differentiable and concave on $[0,1]$, and both incur unit marginal costs of production. Finally assume that players make sequential investments of efforts with player 1 moving first. After player 1 has invested his effort b_1, player 2 observes b_1. A unit prize is allocated to a player who invests larger effort, in case of a tie player 2 receives the prize.

Show that in the Stackelberg equilibrium, conditioned on $v_1 = v$, player 1 is more likely to win than player 2 if and only if $v > 1/F'(F^{-1}(1/2))$.

Show that in the Stackelberg equilibrium, the winning probability of player 1 is always less than or equal to that of player 2.

4

Consider a contest among n players with valuations $v_1 > v_2 > \cdots > v_n > 0$ and unit marginal cost of production. The contest is organized in $1 \leqslant m < n$ stages. In each stage the player who invested the largest effort in that stage wins a unit prize and is no longer eligible for further competition. All losers in a stage except the final one, continue to compete in the next stage. The expected continuation value is discounted with parameter $\delta \in (0,1)$.

Show that in the limit of no discounting ($\delta \uparrow 1$), the winning probabilities x_i of players are given as follows

$$x_i = \begin{cases} 1 - \left(\frac{1}{2}\right)^m \frac{v_{m+1}}{v_1}, & i = 1 \\ 1 - \left(\frac{1}{2}\right)^{m-i+2} \frac{v_{m+1}}{v_i}, & 2 \leqslant i \leqslant m \\ m - \sum_{j=1}^{m} x_j, & i = m+1. \end{cases}$$

5

Given are outcomes of paired comparisons for a set of players $0, 1, \dots, n$ such that there are $m > 0$ comparisons for each pair from the following set of pairs of players

$$(0, 1), (1, 2), \dots, (n-1, n)$$

and the number of times that i won against $i+1$ is $w_{i,i+1} = mp_i$, for given p_i such that $0 < p_i < 1$, for $i = 0, 1, \dots, n-1$. Assume that outcomes are according to the Bradley-Terry model with parameters $\theta_0, \theta_1, \dots, \theta_n$.

Give an explicit characterization of the maximum likelihood estimate of parameters, $\hat{\theta}_0, \hat{\theta}_1, \dots, \hat{\theta}_n$, in terms of given observations $p_0, p_1, \dots, p_{n-1}$. Assume a normalization such that $\hat{\theta}_n = 1$.

END OF PAPER