UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 1:30 pm to 4:30 pm

PAPER 38

ADVANCED FINANCIAL MODELS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $(X_t)_{t \geqslant 0}$ be a discrete-time local martingale with respect to a filtration $(\mathcal{F}_t)_{t \geqslant 0}$. Assume X_0 is not random.

(a) Show that if X_t is integrable for all $t \geqslant 0$, then X is a martingale.

(b) Show that if $X_t \geqslant 0$ almost surely for all $t \geqslant 0$, then X is a martingale.

Let M be a martingale and K be a predictable process, and let $Y_0 = 0$ and

$$Y_t = \sum_{s=1}^{t} K_s (M_s - M_{s-1})$$

for $t \geqslant 1$.

(c) Show that if K is bounded, then Y is a martingale.

(d) Show that in general, Y is a local martingale.

(e) Show that if $Y_T \geqslant 0$ almost surely for a non-random $T > 0$, then $Y_T = 0$ almost surely.

2

Consider a discrete-time market with n-dimensional price process $P = (P_t^{(1)}, \ldots, P_t^{(n)})_{t \geqslant 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{\geqslant 0}$.

(a) What does it mean to say a European contingent claim is attainable? Your answer should include the definition of a self-financing strategy.

Introduce to this market an attainable European contingent claim with payout ξ and maturity date T. Assume that P is a martingale and that the sample space Ω is finite.

(b) Show that the initial replication cost of the claim is $X_0 = \mathbb{E}(\xi)$.

Now suppose the market has $n = 2$ assets. The first asset is cash with constant price $P_t^{(1)} = 1$, and the second is a stock with price $P_t^{(2)} = S_t$. Assume $\mathbb{P}(S_t = S_{t-1}) = 0$ for all $t \geqslant 1$.

(c) Show that the number of shares of stock in the replicating portfolio is given by

$$\pi_t = \frac{\mathrm{Cov}(\xi, S_t | \mathcal{F}_{t-1})}{\mathrm{Var}(S_t | \mathcal{F}_{t-1})}$$

for all $t \geqslant 1$.

(d) Suppose $\xi = g(S_T)$ for a strictly increasing function g. Show that $\pi_T > 0$ almost surely.

3

Consider a market with a bank account with price

$$B_t = e^{\int_0^t r_s \; ds}$$

and a bond with maturity $T > 0$ and price

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds} | \mathcal{F}_t]$$

where $(r_t)_{t \geqslant 0}$ is a given non-negative continuous adapted process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $(\mathcal{F}_t)_{t \geqslant 0}$.

(a) Show that the discounted bond price $(P(t,T)/B_t)_{0 \leqslant t \leqslant T}$ is a $\mathbb{Q}$-martingale.

Let $\mathbb{Q}_T$ be an equivalent measure with density

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{e^{-\int_0^T r_s ds}}{P(0,T)}$$

(b) Show that the discounted bank account $(B_t/P(t,T))_{0 \leqslant t \leqslant T}$ is a $\mathbb{Q}_T$-martingale.

Suppose the spot interest rate r_t evolves as

$$dr_t = r_t(r_t - 1)(dt + dW_t)$$

where W is a $\mathbb{Q}$-Brownian motion and $0 < r_0 < 1$.

(c) Show that there are deterministic functions $A, B : [0, \infty) \to \mathbb{R}$ such that

$$M_t = e^{-\int_0^t r_s ds}(A(T-t) + B(T-t)r_t)$$

is a $\mathbb{Q}$-local martingale. Hence, compute the bond price $P(t,T)$ in terms of r_t.

(d) Compute $\mathbb{E}^{\mathbb{Q}_T}(r_T | \mathcal{F}_t)$ in terms of r_t.

[You may use without proof the fact that $0 \leqslant r_t \leqslant 1$ almost surely for all $t \geqslant 0$.]

4

Consider a one-period market with a stock with initial price $S_0 = 1$ and time-1 price $S_1 > 0$ and a family of call options maturing at $T = 1$ with strikes $K_1 < K_2 < \ldots < K_n$. Let $C(K)$ be the initial price of the option with strike K.

(a) Suppose that $C(K_i) < C(K_{i+1})$ for some i. Exhibit an arbitrage strategy.

(b) Suppose $K_i = \frac{1}{2}(K_{i-1} + K_{i+1})$ for some i and that

$$C(K_i) > \frac{1}{2}[C(K_{i-1}) + C(K_{i+1})].$$

Exhibit an arbitrage strategy.

(c) Suppose

$$C(K) = (1 + K^p)^{1/p} - K.$$

Show that, if there is no arbitrage, then $p > 1$.

(d) Suppose that there is no arbitrage in the market. We now introduce a contingent claim with payout $g(S_1)$. Exhibit a function f (depending on p) such that if the initial price of the claim is

$$\int_0^\infty g(u)f(u)du$$

then the augmented market has no arbitrage.

[Fundamental theorems of asset pricing can be used without proof if clearly stated.]

5

Let W be a Brownian motion and and let $(Z_t)_{t\geqslant 0}$ evolve according to

$$dZ_t = aZ_t \, dt + b \, dW_t, \quad Z_0 = z$$

for constants a, b and z.

(a) Show that for all $t \geqslant 0$, the distribution of the random variable Z_t is given by

$$Z_t \sim N\left(e^{at}z, \frac{b^2}{2a}(e^{2at} - 1)\right).$$

Consider a market consisting of a bank account and a stock with corresponding price dynamics

$$\begin{aligned} dB_t &= B_t r \, dt \\ dS_t &= \mu \, dt + \sigma \, dW_t \end{aligned}$$

where r, μ and σ are positive constants, and W is a Brownian motion generating the filtration. (Note that this is *not* the Black–Scholes model.)

(b) Show that there exists a trading strategy which replicates the payout of a European call option with strike K and maturity T in such a way that the value of the portfolio is always non-negative. What is the minimum amount of capital needed to finance this replication strategy?

(c) Show that the number of shares of stock in the replicating portfolio is always non-negative but never greater than one.

[You may use results from the lectures without proof if they are clearly stated. Also, if you need to invoke Girsanov's theorem, you may assume that the appropriate hypotheses are satisfied.]

6

Let $(P_t)_{t \geqslant 0}$ be a n-dimensional Itô process modelling the prices of n assets.

(a) Explain why a self-financing investor's time-t wealth X_t satisfies the two equations

$$X_t = H_t \cdot P_t$$
$$dX_t = H_t \cdot dP_t - c_t dt$$

where H_t is the investor's portfolio and c_t is his consumption rate at time t.

(b) Let $(Y_t)_{t \geqslant 0}$ be a state price density. Show that

$$d(Y_t X_t) = H_t \cdot (Y_t P_t) - Y_t c_t \; dt$$

(c) Assume that the pair (H, c) are such that the investor's wealth is always non-negative. Using standard results from stochastic calculus, show that the stochastic integral

$$M_t = \int_0^t H_s \cdot d(Y_s P_s)$$

defines a supermartingale.

(d) Prove that

$$\mathbb{E}\left(\int_0^\infty Y_t c_t \; dt \right) \leqslant Y_0 X_0.$$

(e) Let $U : [0, \infty) \to \mathbb{R}$ be a strictly increasing, strictly concave function bounded from above. Suppose that $e^{-bt} U'(c_t) = Y_t$ for all $t \geqslant 0$ and that the inequality in part (d) is actually equality. Let $(\hat{H}, \hat{c})$ be another strategy with corresponding non-negative wealth process $\hat{X}$. Assuming $\hat{X}_0 = X_0$, show that

$$\mathbb{E}\left(\int_0^\infty e^{-bt} U(c_t) dt \right) \geqslant \mathbb{E}\left(\int_0^\infty e^{-bt} U(\hat{c}_t) dt \right)$$

END OF PAPER