UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 4 June, 2014 9:00 am to 12:00 pm

PAPER 37

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the problem to

$$\begin{array}{ll} \text{maximize} & \dfrac{c^T x + \alpha}{d^T x + \beta} \\ \text{subject to} & Ax \leqslant b \end{array}$$

for $x \in \mathbb{R}^n$ and given $c, d \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

(a) Assume that the set $\{x : Ax \leqslant b\}$ is non-empty and bounded and that $d^T x^* + \beta > 0$ for some optimal solution x^*. Prove that an optimal solution can be found by solving the following linear program:

$$\begin{array}{ll} \text{maximize} & c^T y + \alpha t \\ \text{subject to} & Ay \leqslant bt \\ & d^T y + \beta t = 1 \\ & t \geqslant 0. \end{array}$$

(b) Solve the problem for

$$c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad d = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \alpha = 0, \quad \beta = 2,$$
$$A = \begin{pmatrix} 2 & 3 \\ 1 & -2 \\ -3 & 1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix}.$$

2

For $\epsilon \in \mathbb{R}^3$, let $\phi(\epsilon)$ denote the optimum objective value of the problem to

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 + 3x_3 \\ \text{subject to} & x_1 + 2x_2 + 2x_3 \leqslant 6 + \epsilon_1 \\ & 3x_1 + 2x_2 + x_3 \leqslant 5 + \epsilon_2 \\ & 2x_1 + 2x_2 + 4x_3 \leqslant 4 + \epsilon_3 \\ & x_1, x_2, x_3 \geqslant 0. \end{array}$$

(a) Determine $\phi((0,0,0)^T)$ and prove from first principles that the value you have found is correct.

(b) Derive an expression for $\phi(\epsilon)$ when the entries of ϵ have small absolute values. For which values of ϵ_3 is the expression correct when $\epsilon_1 = \epsilon_2 = 0$? Justify your answers.

3

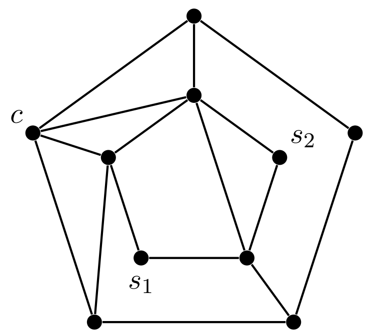
Consider an undirected graph (V, E) representing a communication network between a single client $c \in V$ and a set $S \subseteq V$ of servers. The client is connected to the servers via a set of intermediary nodes, represented by the vertices in $V \setminus (\{c\} \cup S)$, and via a set of links, represented by the edges in E. The client can connect to a particular server $s \in S$ if there exists a path from c to s in (V, E). Assume further that links are subject to failure, and call a network k-link-safe if the client can still connect to at least one server after the simultaneous failure of an arbitrary set of k links.

(a) Describe a polynomial-time algorithm that finds the maximum value k for which a given network is k-link-safe. Explain carefully why the algorithm is correct and requires only a polynomial number of steps.

Now assume that intermediary nodes can fail as well, and call a network m-safe if the client can still connect to at least one server after the simultaneous failure of an arbitrary set of m links and intermediary nodes.

(b) Explain how the above algorithm can be extended to decide whether a network is m-safe, for a given value of m.

(c) Find the maximum values k and m for which the following network with client c and servers s_1 and s_2 is k-link-safe and m-safe, and prove that these values are correct:



4

Imagine a situation where each of two players chooses a number from the set $\{1, 2, 4\}$ and correspondingly bets an amount of £1, £2, or £4. This is done without knowledge of the other player's choice. If the two numbers chosen are identical, the amounts are returned to the respective players. If the two numbers differ by 1, the first player receives the whole amount. Otherwise the second player receives the whole amount.

(a) Model this situation as a normal-form game.

(b) Explain why it might not be a good idea to apply the Lemke-Howson algorithm to the game. Show that its equilibria can instead be found by solving the problem to

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 + x_3 \\ \text{subject to} & 5x_1 + 6x_2 + x_3 \geqslant 1 \\ & 7x_1 + 5x_2 + x_3 \geqslant 1 \\ & 4x_1 + 3x_2 + 5x_3 \geqslant 1 \\ & x_1, x_2, x_3 \geqslant 0. \end{array}$$

(c) Solve this problem using the simplex method and find all equilibria of the game. You may for example start from the basic feasible solution where $x_1 = \frac{1}{4}$ and $x_2 = x_3 = 0$, pivoting appropriately to bring the initial tableau into the right form.

Assume now that players get the chance to double their stakes, i.e., that each player not only chooses a number from the set $\{1, 2, 4\}$ but also decides whether the corresponding bets are £1, £2, and £4, or £2, £4, and £8. The outcome of the game is determined as before.

(d) Should the players double their stakes? Justify your answer.

5

Call a coalitional game (N, v) a *weighted voting game* if there exist $w : N \to \mathbb{N}$ and $t \in \mathbb{N}$ such that for all $S \subseteq N$, $v(S) = 1$ if $\sum_{i \in S} w(i) > t$ and $v(S) = 0$ otherwise.

(a) The United Nations Security Council has fifteen members, five of which are permanent. Resolutions require the positive vote of at least nine members to pass, but can be vetoed by any permanent member. Define a weighted voting game (N, v) such that $v(S) = 1$ if and only if a resolution supported by the members of $S \subseteq N$ passes.

(b) Compute the vector of Shapley values for this game.

Recall that a coalitional game (N, v) is convex if $v(S \cup T) \geqslant v(S) + v(T) - v(S \cap T)$ for all $S, T \subseteq N$.

(c) Prove or disprove that every weighted voting game is convex.

(d) Prove or disprove that in every convex game, the vector of Shapley values is contained in the core.

6

Consider the problem MAX-2SAT of finding, for a given Boolean formula in conjunctive normal form with *at most two* literals per clause, the maximum number of clauses that can be satisfied simultaneously.

(a) Let a, b, c, and d be Boolean variables and denote their respective negations by $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$. Show that the formula $(a \vee b \vee c)$ is satisfied if and only if there exists a value for d such that 7 of the following formulae are satisfied: (a), (b), (c), (d), $(\bar{a} \vee \bar{b})$, $(\bar{a} \vee \bar{c})$, $(\bar{b} \vee \bar{c})$, $(a \vee \bar{d})$, $(b \vee \bar{d})$, and $(c \vee \bar{d})$.

(b) Show that it is NP-complete to decide, for a given formula with at most two literals per clause and a given value k, whether k clauses of the formula can be satisfied simultaneously. You may assume NP-completeness of the satisfiability problem for formulae with three literals per clause.

(c) Show that for any formula with at most two literals per clause, a random assignment satisfies at least half of the clauses in expectation. Use the method of conditional probabilities to obtain a polynomial-time 1/2-approximation algorithm for the MAX-2SAT problem.

(d) Explain how this result can be improved if each variable appears at most once in a clause of size one, by ensuring that such a clause is satisfied with probability $p > 1/2$. Why is $p = \frac{\sqrt{5}-1}{2}$ a good choice and which approximation ratio does it lead to?

END OF PAPER