UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 30 May, 2014 9:00 am to 11:00 am

PAPER 36

TIME SERIES AND MONTE CARLO INFERENCE

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Time Series

Series 1 and 2 are realisations of two time series models, each containing 1000 observations. The first 100 observations and the autocorrelation functions (ACF) based on all observations are shown in Figure 1.

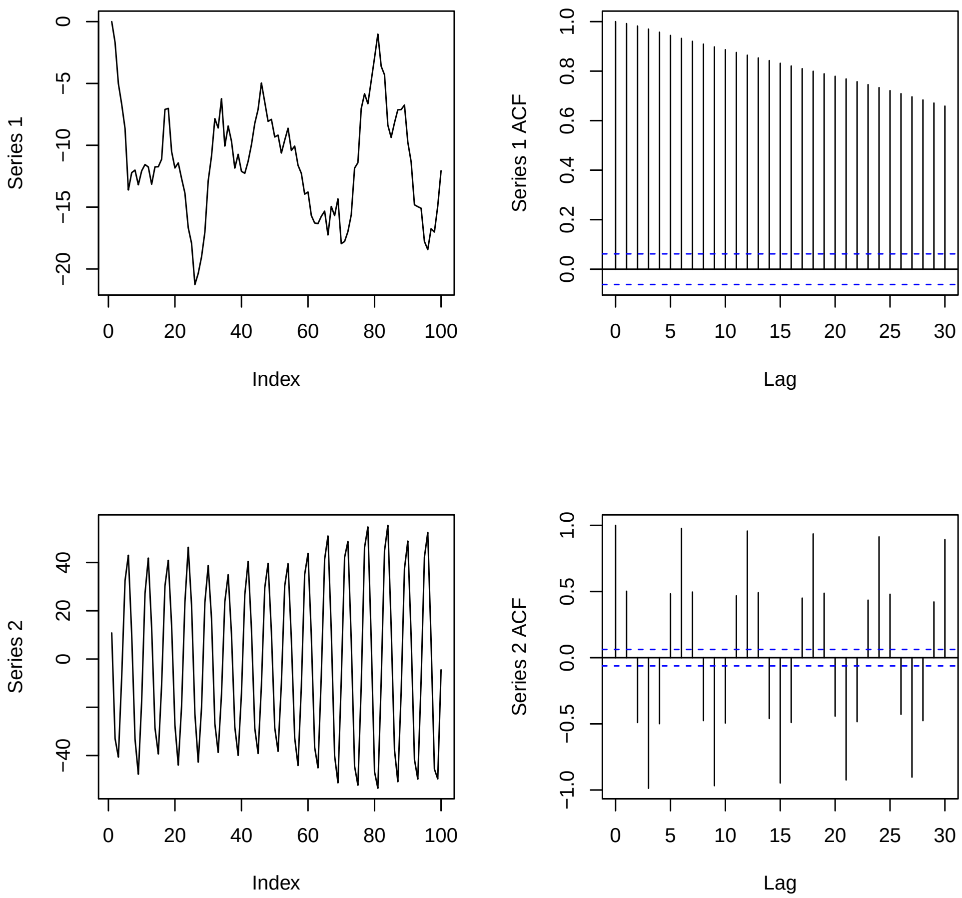

Figure 1: Time series and the ACFs

(a) Discuss the non-stationarity of these two time series and the zeros of the autoregressive polynomials.

(b) Suitable differencing is applied to Series 1 and let the resulting differenced series be denoted $\{X_t\}$. Three ARMA(p,q) models were fitted to $\{X_t\}$ and the fitted results are given below. (Estimators and standard errors of the coefficients are rounded to two decimal places; sigma squared estimators are rounded to 1 significant figure.) Which is your preferred model of $\{X_t\}$? Give your reasons, and write down the corresponding model for $\{X_t\}$ and spectral density thereof.

ARMA(2,2)

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Call:
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`arima(x = series.1.diff, order = c(2, 0, 2), include.mean = F)`

`Coefficients:`

	ar1	ar2	ma1	ma2
	-0.20	0.31	0.38	-0.06
s.e.	0.14	0.12	0.14	0.12

`sigma squared estimated as 4:  log likelihood= -2142.99, aic = 4295.97`

ARMA(2,1)

`Call:`

`arima(x = series.1.diff, order = c(2, 0, 1), include.mean = F)`

`Coefficients:`

	ar1	ar2	ma1
	-0.30	0.28	0.50
s.e.	0.16	0.04	0.16

`sigma squared estimated as 4:  log likelihood= -2143.13, aic = 4294.26`

ARMA(1,1)

`Call:`

`arima(x = series.1.diff, order = c(1, 0, 1), include.mean = F)`

`Coefficients:`

	ar1	ma1
	0.62	-0.42
s.e.	0.07	0.08

`sigma squared estimated as 4:  log likelihood= -2152.9, aic = 4311.81`

(c) Adopting the model chosen in your answer to part (b) above, discuss the causality and invertibility of $\{X_t\}$. Find polynomials $\tilde{\phi}(z)$ and $\tilde{\theta}(z)$ such that

$$\tilde{\phi}(B)X_t = \tilde{\theta}(B)W_t,$$

where $\{W_t\} \sim \mathrm{WN}(0,1)$ and B is the backward shift operator.

(d) Derive the linear process representation (MA(∞)) of the $\{X_t\}$ (calculate the first 5 coefficients and give the formula for the remaining coefficients) with the parameters given in (b). Derive the autocovariance function (ACVF) and ACF of this linear process representation in terms of the MA(∞) coefficients.

2 Time Series

(a) Point out the mistake in the following statement, write down the missing condition and explain why the statement is wrong without that condition.

$\{X_t\}$ *is the stationary solution of*

$$X_t = \phi X_{t-1} + Z_t, \quad t = 0, \pm 1, \ldots, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma_z^2).$$

(b) Write down the proper condition such that $\{X_t\}$ is a causal function of $\{Z_t\}$. Let $\{Y_t\}$ be the AR(1) plus noise series defined by

$$Y_t = X_t + W_t, \quad \{W_t\} \sim \mathrm{WN}(0, \sigma_w^2),$$

and $E(W_s Z_t) = 0$ for all s and t. Show that $\{Y_t\}$ is stationary and find its autocovariance function (ACVF).

(c) Under the conditions in (b), show that $\{Y_t\}$ is an ARMA(1,1) process and express the three parameters of this model in terms of ϕ, σ_w^2 and σ_z^2.

(d) Write down one **complete** state-space representation of $\{Y_t\}$ and specify the observation and state equations.

3 Monte Carlo Inference

Let $X_0 = X_1 = 0$. We consider the following AR(2) model defined recursively for any $t \in \mathbb{N}$:

$$X_{t+2} = aX_{t+1} + bX_t + \epsilon_t,$$

where the ϵ_t are i.i.d. Gaussian of mean 0 and variance 1 (we write $\mathcal{N}(0,1)$ for this distribution). We observe the chain $X_1, \ldots, X_n$ until some time $n \geqslant 2$. We would like to estimate the parameters (a, b) *using Bayesian inference.*

(a) Explain how to generate observations from a Gaussian distribution $\mathcal{N}(0,1)$ of mean 0 and variance 1. Prove this method works.

(b) Write the density of $X_2|X_1$ (i.e. the density of X_2 knowing X_1), the density of $X_3|(X_2, X_1)$, and the density of $X_4|(X_3, X_2, X_1)$. Deduce in a similar way the distribution of $X_{t+2}|(X_{t+1}, X_t, \ldots, X_1)$ for $t \geqslant 2$.

(c) Deduce from your answer to the last question the likelihood of the samples $X_1, \ldots, X_n$.

(d) In order to apply Bayesian inference to this chain, we set priors for (a, b). We choose $a \sim \mathcal{N}(0,1)$ and $b \sim \mathcal{N}(0,1)$, with a, b independent of each other. What is the posterior density $\pi(a, b)$ of (a, b) knowing $X_1, \ldots, X_n$, that is to say the density of

$$(a, b)|X_1, \ldots, X_n?$$

What are the conditional distributions, of

$$b|a, X_1, \ldots, X_n, \quad \text{and} \quad a|b, X_1, \ldots, X_n?$$

[*Hint: Consider a bivariate normal distribution* $\mathcal{N}(\mu, \Sigma)$ *where* $\mu = (\mu_i)_{i\in\{1,2\}}$ *and* $\Sigma = (\Sigma_{i,j})_{(i,j)\in\{1,2\}^2}$. *Consider* $Y = (Y_1, Y_2) \sim \mathcal{N}(\mu, \Sigma)$. *Then the conditional distribution of* Y_1 *knowing* $Y_2 = y_2$ *is* $\mathcal{N}\Big(\mu_1 + \frac{\Sigma_{1,2}}{\Sigma_{2,2}}(y_2 - \mu_2), \Sigma_{1,1} - \frac{\Sigma_{1,2}}{\Sigma_{2,2}}\Sigma_{2,1}\Big)$ *(and the same goes for the conditional distribution of* Y_2 *knowing* Y_1 *changing the index accordingly).*]

(e) Explain how you can implement the Gibbs sampler using these posterior conditional distributions. You are given a function $\phi : \mathbb{R}^2 \to \mathbb{R}$. How would you estimate $\phi(a, b)$ using the output of the Gibbs sampler?

4 Monte Carlo Inference

Consider two densities f and g defined on $\mathbb{R}$, with associated measures fdx and gdx, for which there exists a constant $M > 0$ such that $\forall x \in \mathbb{R}$, we have $f(x) \leqslant Mg(x)$. Assume that you can generate random variables distributed according to gdx with your computer. Let ϕ be a function such that $\int_{\mathbb{R}} \phi(x)^2 f(x)dx < \infty$.

(a) Explain how you can make an estimate, $\hat{\theta}_1$, for the quantity $\theta = \int_{\mathbb{R}} \phi(x)f(x)dx$ by using importance sampling. Compute the mean and variance of $\hat{\theta}_1$, proving that they are both finite. What is the limiting distribution of $\sqrt{n}(\hat{\theta}_1 - \theta)$? Justify your answer.

(b) Explain how you can generate a random variable with density f using observations generated according to g. Prove why your method works.

(c) Assume that you have a dataset of n observations $X_1, \ldots, X_n$ distributed according to g. You use the technique of the previous question in order to transform this sample set of i.i.d. random variables $X_1, \ldots, X_n$ distributed according to gdx into a sample set of i.i.d. random variables $Y_1, \ldots, Y_N$ distributed according to fdx, where N is a random variable. What is its distribution? What are the expectation and the variance of N? Justify your answers.

(d) What is the limiting distribution of $\frac{N - \mathbb{E}N}{\sqrt{n}}$? What do you think is the limiting distribution of $\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N} \phi(Y_i) - \theta\right)$? [You need not prove your assertions.]

(e) Which estimate of θ should be preferred, the importance sampling estimate $\hat{\theta}_1$ or the estimate $\hat{\theta}_2 = \frac{1}{N}\sum_{n=1}^{N} \phi(Y_i)$? [You need not prove your assertions.]

END OF PAPER