UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 2 June, 2014 1:30 pm to 3:30 pm

PAPER 31

ACTUARIAL STATISTICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Suppose that the total amount claimed in one time period on a particular risk is a random sum $S = X_1 + \cdots + X_N$ ($S = 0$ if $N = 0$), where the claim amounts $X_1, X_2, \ldots$ are independent identically distributed positive random variables with moment generating function M_X and the number of claims N is independent of the claim amounts and has probability generating function G_N. Show that the moment generating function $M_S(r)$ of S is given by $M_S(r) = G_N\big(M_X(r)\big)$.

(b) For another risk, suppose that with probability $1-q$ ($0 \leqslant q \leqslant 1$) there are no claims in one time period and with probability q the amount claimed in the time period is a positive random variable Z. Let T be the total amount claimed in one time period for this risk, so that

$$T = \begin{cases} 0 & \text{with probability } 1-q \\ Z & \text{with probability } q. \end{cases} \tag{1}$$

Write down an expression for the moment generating function $M_T(r)$ of T in terms of q and the moment generating function of Z.

By writing $G_N(z) = p + (1-p)G_{\tilde{N}}(z)$ where p is in $[0,1]$ and $G_{\tilde{N}}$ is the probability generating function of another random variable $\tilde{N}$, show that S in (a) may be written in the form of (1) for some q and Z where you should identify q and show that Z is itself a random sum.

When the X_i are exponentially distributed with mean μ and N has probability mass function given by $\mathbb{P}(N = n) = 0.5^{n+1}$, $n = 0, 1, 2, \ldots$, find q and the distribution of Z explicitly in terms of μ. Hence write down the distribution function of S.

2

(a) In an insurance portfolio, claims arrive in a Poisson process, the claim sizes $X_1, X_2, \ldots$ are independent identically distributed positive random variables with mean μ ($< \infty$) independent of the claim arrivals process, and the relative safety loading factor is θ (> 0). Let $M(r) = \mathbb{E}(e^{X_1 r})$ be the moment generating function of the claim size distribution and suppose that there exists r_∞, $0 < r_\infty \leqslant \infty$, such that $M(r) < \infty$ for $r < r_\infty$ and $M(r) \uparrow \infty$ as $r \uparrow r_\infty$. Show that there exists a unique R (the adjustment coefficient) in $(0, r_\infty)$ that solves

$$M(r) - 1 = (1+\theta)\mu r.$$

(b) Now suppose that the X_i are exponentially distributed with mean μ.

Find the adjustment coefficient.

After a change in legislation, the company has to incorporate an additional administrative stage to the settlement of each claim. The resulting additional administrative costs are independent exponentially distributed random variables with mean 0.5μ, independent of the X_i. The company keeps the same relative safety loading factor θ. Find the new adjustment coefficient.

If $\theta = 1$, compare the new adjustment coefficient with the adjustment coefficient for the portfolio without the additional administrative costs.

3

In a classical risk model with positive safety loading, suppose that the claim arrival rate is λ (> 0), the premium rate is c (> 0), and the claim sizes have density function f, distribution function F and mean μ. Let $\psi(u)$ be the probability of ruin with initial capital u ($\geqslant 0$), and let $\varphi(u) = 1 - \psi(u)$ be the survival probability. Prove that

$$\varphi(u) = 1 - \frac{\lambda\mu}{c} + \frac{\lambda}{c}\int_0^u \varphi(u-x)\big(1 - F(x)\big)\,dx.$$

[*Hint: You may assume that* $\psi(0) = \frac{\lambda\mu}{c}$.]

A company has n independent portfolios. The surplus process for portfolio i, $i = 1, \ldots, n$ is modelled as the surplus process for a classical risk model with claim arrival rate λ_i per accounting period, income rate c_i per accounting period, initial capital $u_i = 0$, and claim sizes that are exponentially distributed with mean μ_i. Find the probability that all n portfolios survive.

The company decides to combine these n portfolios into one merged portfolio. Show that the total amount claimed on the merged portfolio in one accounting period has a compound Poisson distribution with Poisson parameter $\lambda = \sum_{i=1}^n \lambda_i$ and "claim-size" density

$$f(x) = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda\mu_i} e^{-x/\mu_i}, \quad x > 0.$$

Assuming that the surplus process of the merged portfolio may be modelled as the surplus process of a classical risk model with arrival rate λ and "claim size" density f, find, in terms of relevant quantities for the n original portfolios, the probability that the merged portfolio survives.

4

A portfolio consists of m_j individual risks in year j, $j = 1, \ldots, n+1$, where the m_j are known. Suppose that, given a risk parameter θ, the number of claims on the individual risks in year j are independent with mean $\mu(\theta)$ and variance $\sigma^2(\theta)$. In year j, let Y_j be the number of claims on the whole portfolio and let $X_j = Y_j/m_j$ be the average number of claims per individual risk. Show that $\mathbb{E}\big[X_j \mid \theta\big] = \mu(\theta)$ and $\text{var}\big[X_j \mid \theta\big] = \sigma^2(\theta)/m_j$.

Assume that, conditional on θ, the Y_j are independent and that $Y_1 = y_1, \ldots, Y_n = y_n$ are observed. Show that the Bühlmann–Straub credibility estimate for the expected number of claims on the whole portfolio in year $n+1$ is

$$m_{n+1}\left(Z\frac{\sum_{j=1}^n y_j}{\sum_{j=1}^n m_j} + (1-Z)\mathbb{E}\big[\mu(\theta)\big]\right),$$

where you should specify Z.

Suppose that, given θ, the number of claims on an individual risk has a Poisson distribution with mean θ, and that *a priori* θ has a gamma distribution with mean α/β and variance α/β^2.

(a) Find the Bühlmann–Straub credibility estimate for the expected number of claims on the whole portfolio in year $n+1$.

(b) Show that the Bayesian estimate of θ under quadratic loss is

$$\frac{\alpha + \sum_{j=1}^n y_j}{\beta + \sum_{j=1}^n m_j}.$$

Write down the resulting estimate for the expected number of claims on the whole portfolio in year $n+1$. Compare your answer with the credibility estimate in (a).

END OF PAPER