UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 6 June, 2014 1:30 pm to 4:30 pm

PAPER 3

REPRESENTATION THEORY AND QUIVERS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

Throughout, let k be an algebraically closed field.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define a quiver Q. What is a representation X of Q? Define a morphism $\theta : X \to Y$ between two representations X and Y of Q. Define the path algebra, $A := kQ$ of Q.

Denote by $\mathrm{Rep}(Q)$ the category of representations of Q. Show that $\mathrm{Rep}(Q)$ is equivalent to the module category Mod A. State a necessary and sufficient condition on Q for A to be finite-dimensional over k.

Construct (up to isomorphism) all the indecomposable representations of the quiver A_3 consisting of three vertices and two arrows between them. You may choose between either of the two possible orientations:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{or} \quad \bullet \longrightarrow \bullet \longleftarrow \bullet$$

You should justify your answer. [Do not attempt the classification of indecomposables of both quivers and do not appeal to Gabriel's theorem.]

2

Let Q be a quiver having no oriented cycle. Given $i \in Q_0$ define the simple module $S(i)$ with dimension vector ε_i (the ith basis vector of $\mathbb{Z}^{Q_0}$). Show that every simple representation of Q is isomorphic to $S(i)$ for a unique $i \in Q_0$. Show also that any finite-dimensional semisimple representation is uniquely determined by its dimension vector, up to isomorphism.

Now let Q be arbitrary. Decompose $1 = \sum_{i \in Q_0} e_i$ into orthogonal idempotents, and in the corresponding decomposition of kQ let $P(i) = P(e_i) = kQe_i$ be the ith summand. Characterise both $P(i)$ and its endomorphism algebra $\mathrm{End}_Q(P(i))$ in terms of spans of certain paths. If i is any vertex and $X \in \mathrm{Rep}(Q)$, show that $\mathrm{Hom}_Q(P(i), X) \cong X_i$. Deduce that $P(i)$ is indecomposable.

Consider the representation $X : k \xrightarrow{1} k \longleftarrow 0$ of the quiver A_3. Show that X is projective.

3

Let V, W be representations of the quiver Q. Define $\mathrm{Ext}^1_Q(V,W)$. With the usual notation, consider the map

$$\gamma_{V,W} : \bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, W_i) \longrightarrow \bigoplus_{\rho \in Q_1} \mathrm{Hom}(V_{s(\rho)}, W_{t(\rho)})$$

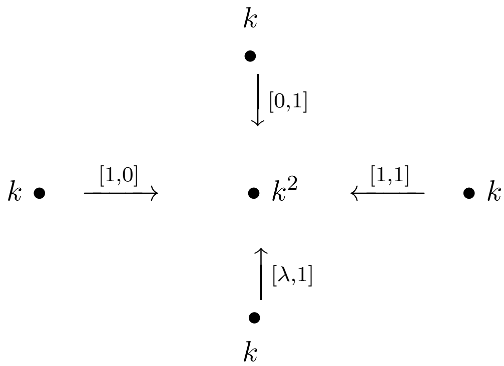
$$(u_i)_{i \in Q_0} \longmapsto \big(u_{t(\rho)} f_\rho - g_\rho u_{s(\rho)}\big)_{\rho \in Q_1}.$$

Identify (without justification) the kernel and cokernel of $\gamma_{V,W}$. Assuming V, W are finite-dimensional, define the Ringel form on $\mathbb{R}^{Q_0}$ for V, W. Explain why your definition depends only on the dimension vectors of V, W.

Compute $\dim \mathrm{Ext}^1_Q(V,V)$ in each of the following situations.

(a) $Q = L = \overset{\bullet}{\circlearrowright}$, the loop, and $V = k$ with transformation given by the matrix $[\lambda]$ (for $\lambda \in k$).

(b) $Q = S_4$, the four-subspace quiver, and, for $\lambda \in k$, V is as follows:



4

What does it mean to say that an affine algebraic group is unipotent?

Let $x = (x_\rho)_{\rho: i \to j} \in \mathrm{Rep}_Q(\underline{n})$ correspond to the finite-dimensional representation $X = X_x$ of the quiver Q. Explain briefly why the linear algebraic group $\mathrm{Aut}_Q(X)$ is open in $\mathrm{End}_Q(X)$. Prove that $\mathrm{Aut}_Q(X)$ decomposes as a semi-direct product of a closed, normal, unipotent subgroup and a product of general linear groups. Deduce a criterion for the indecomposability of a representation in terms of its automorphism group.

Now recall the existence of a 4-term exact sequence of k-spaces

$$0 \to \mathrm{End}_Q(X) \to \bigoplus \mathrm{End}_k(X_i) \xrightarrow{\xi_x} \mathrm{Rep}_Q(\underline{n}) \to \mathrm{Ext}^1_Q(X, X).$$

You are not being asked to prove existence but you should state explicitly a formula for ξ_x.

Find the differential at the identity of the orbit map associated to X. By identifying $\mathrm{Rep}_Q(\underline{n})$ with its own tangent space, show that the image of ξ_x is the Zariski tangent space $T_x(\mathcal{O}_X)$ (standard dimension formulae can be quoted without proof).

Deduce that $\mathcal{O}_X$ is open in $\mathrm{Rep}_Q(\underline{n})$ if and only if $\mathrm{Ext}^1_Q(X, X) = 0$, in which case it is the unique dense orbit of dimension vector $\underline{n}$ (standard facts about dimensions of normal spaces may be assumed).

5

Let A be a finitely-generated algebra. Define $\mathrm{Rep}_A(\underline{n})$ for $\underline{n} \in \mathbb{N}^m$. Explain how $\mathrm{GL}(\underline{n})$ acts on this affine variety. Let $\mathcal{O}_X$ be the orbit in $\mathrm{Rep}_A(\underline{n})$ of points x with $X_x \cong X$. Prove that

$$\dim \mathrm{GL}(\underline{n}) - \dim \mathcal{O}_X = \dim \mathrm{Stab}(x) = \dim \mathrm{Aut}(X) = \dim \mathrm{End}(X).$$

What does it mean to say that X degenerates to Y? For the special case of $\mathrm{Rep}_A(r)$ for $r \in \mathbb{N}$, show directly that if

$$0 \to X' \to X \to X'' \to 0$$

is an exact sequence of finite-dimensional representations then X degenerates to $X' \oplus X''$. [*Hint: An element $x \in Rep_Q(r)$ is defined by matrices $x_g \in M_r(k)$ where g runs through a set of generators of A.*]

Assuming the above result about degeneration holds for $\mathrm{Rep}_A(\underline{n})$, where $A = kQ$, the path algebra of a quiver, deduce that the finite-dimensional representation X of Q degenerates to the associated graded representation $\mathrm{gr}(X)$ defined by a composition series of X.

(i) Determine $\mathrm{Rep}_Q(\underline{n})$ when Q is the quiver

$$\bullet \xrightarrow{\rho} \bullet$$

Determine the orbits in $\mathrm{Rep}_Q(\underline{n})$.

(ii) Let Q be the quiver

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

Let $\underline{n} = (2, 2, 2)$ and $X = k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\mathrm{id}} k^2$. Find $\mathcal{O}_X$ and show that it is open in $\mathrm{Rep}_Q(\underline{n})$ (hence $\overline{\mathcal{O}_X} = \mathrm{Rep}_Q(\underline{n})$). Determine the orbits on the boundary. [Recall that the boundary of a space T is the set of points in $\overline{T}$ not belonging to the interior of T.]

6

What does it mean to say that a (finite-dimensional) representation X of the quiver Q is a brick? Give an example of an indecomposable module for a quiver which is not a brick (only brief justification is required).

Which of the following are bricks (justifying briefly your answer)?

(i) An arbitrary indecomposable module for the quiver $\bullet \longrightarrow \bullet$.

(ii) The Kronecker quiver K_2 and the representation X is $k^2 \overset{\rho_1}{\underset{\rho_2}{\rightrightarrows}} k^2$ where $\rho_1 =$id and $\rho_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Show that if X is indecomposable and not a brick, then X contains a brick Y such that $\mathrm{Ext}^1(Y, Y) \neq 0$.

Now let q be the Tits form and suppose it is positive definite. Show that every indecomposable X is a brick with no self-extensions. Show also that if $\underline{n} = \underline{\dim} X$, then $q(\underline{n}) = 1$.

END OF PAPER