UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 5 June, 2014 9:00 am to 11:00 am

PAPER 29

SCHRAMM–LOEWNER EVOLUTIONS

*There are **FOUR** questions in total.*

The questions carry equal weight.

*Attempt **THREE** questions.*

*If you submit answers to **FOUR** questions, then your overall mark will be determined by the best **THREE** answers you submit.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What is meant by a compact $\mathbb{H}$-hull?

What does it mean to say that a random family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ is a Schramm–Loewner evolution?

Show that, if $(K_t)_{t\geqslant 0}$ is a Schramm–Loewner evolution, then $(K_t)_{t\geqslant 0}$ is scale-invariant and has the domain Markov property. If you rely on any statements identifying Loewner transforms, these should be justified.

Show that, moreover, these properties characterize Schramm–Loewner evolutions, in a sense which you should make precise.

2

Let $\alpha \in (0, 2\pi]$ and introduce the open cone $W_\alpha := \{re^{i\theta},\ r > 0,\ \theta \in (-\alpha/2, \alpha/2)\}$. Let B be a complex Brownian motion started at 1. For $r > 1$, denote by $T(r)$ the first exit time of the ball of radius r centered at 0 i.e. $T(r) := \inf\{t \geqslant 0,\ |B_t| = r\}$. We would like to compute the probability that the complex Brownian motion exits the disk before leaving the cone i.e. $\mathbb{P}_1(B[0, T(r)] \subseteq W_\alpha)$

a) Show that we can reduce the problem to the case $\alpha = \pi$ thanks to the map $z \mapsto z^{\pi/\alpha}$.

We fix now $\alpha = \pi$. Denote by $S := \inf\{t \geqslant 0,\ \mathrm{Re}(B_t) = 0\}$ the first hitting time of the imaginary axis by the Brownian motion.

b) Using a reflection on the imaginary axis, prove that

$$\mathbb{P}_1(T(r) < S) = \mathbb{P}_1(\mathrm{Re}(B(T(r))) > 0) - \mathbb{P}_1(\mathrm{Re}(B(T(r))) < 0).$$

c) With a proper scaling and Möbius transformation, deduce that

$$\mathbb{P}_1(T(r) < S) = \frac{2}{\pi}\,\mathrm{Arctan}\left(\frac{2r}{r^2 - 1}\right)$$

[You can use the formula:

$$\int_{-\pi/2}^{\pi/2} \frac{r^2 - 1}{1 + r^2 - 2r\cos(\theta)} d\theta = \pi + 2\mathrm{Arctan}\left(\frac{2r}{r^2 - 1}\right)]$$

d) Write down the equivalent formula for an arbitrary α.

3

a) Let K be a compact $\mathbb{H}$-hull. What is the definition of the mapping-out function? Show briefly that it is indeed uniquely defined (you can use the results of the lectures about conformal automorphism without proof).

We now denote by g_K the mapping-out function of K.

b) Write down the definition of the half-plane capacity and its characterisation in terms of the Brownian motion (without proof).

c) By using for example the extension property of conformal isomorphism, justify that the mapping-out function g_K is locally bounded and deduce that the map $z \mapsto g_K(z) - z$ is uniformly bounded.

d) Justifying carefully your arguments, prove the inequality $\mathrm{Im}(g_K(z)) \leqslant \mathrm{Im}(z)$.

e) Let $\tau_R := \inf\{t \geqslant 0 : \mathrm{Im}(B_t) = R \text{ or } B_t \notin H\}$. Show that

$$\mathrm{Im}(g_K(z)) = \lim_{R \to +\infty} R\, \mathbb{P}(\mathrm{Im}(B_{\tau_R}) = R).$$

4

Let $(\gamma_t)_{t \geqslant 0}$ be an SLE(4) path and write $\gamma^* = \{\gamma_t : t \in (0, \infty)\}$. Fix $z \in \mathbb{H}$ and set $Z_t = g_t(z) - \xi_t$, where $(g_t)_{t \geqslant 0}$ and $(\xi_t)_{t \geqslant 0}$ are the associated Loewner flow and transform respectively. By considering a suitable transformation of the process $(Z_t)_{t \geqslant 0}$, show that $z \notin \gamma^*$ almost surely.

Show further that the process $(\arg(Z_t))_{t \geqslant 0}$ is a martingale.

It is known that $(\gamma_t)_{t \geqslant 0}$ is a simple path and $\gamma_t \to \infty$ as $t \to \infty$ almost surely. Find the probability that z is in the left connected component of $\mathbb{H} \setminus \gamma^*$.

END OF PAPER