UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 1:30 pm to 3:30 pm

PAPER 28

PERCOLATION AND RELATED TOPICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the bond percolation model on the cubic lattice $\mathbb{Z}^d$ where $d \geqslant 2$. Define the critical probability $p_c(d)$ and the connective constant $\mu(d)$ of $\mathbb{Z}^d$.

Show that

$$\frac{1}{\mu(d)} \leqslant p_c(d) \leqslant 1 - \frac{1}{\mu(2)}.$$

Deduce that

$$\frac{1}{2d-1} \leqslant p_c(d) \leqslant \frac{2}{3}.$$

2

Let E be a finite set and $\Omega = \{0,1\}^E$. What does it mean to say that a subset $A \subseteq \Omega$ is *increasing*?

State the Harris–FKG inequality for positive association, and the BK inequality for disjoint occurrence, in the case of two increasing events A and B. You should explain any further notation that you introduce.

Consider bond percolation on $\mathbb{Z}^d$ where $d \geqslant 2$. Let $\Lambda_k = [-k,k]^d$ and $\partial\Lambda_k = \Lambda_k \setminus \Lambda_{k-1}$, and let $g_k = P_p(0 \leftrightarrow \partial\Lambda_k)$. Show that

$$g_n \leqslant g_{n-m} \sum_{y \in \partial\Lambda_m} P_p(0 \leftrightarrow y), \qquad 1 \leqslant m \leqslant n.$$

Let $\chi(p)$ be the mean number of vertices joined to 0 by open paths, and let p be such that $\chi(p) < \infty$. Show that there exists $\mu = \mu(p) > 0$ such that $g_k \leqslant e^{-\mu k}$ for $k \geqslant 1$.

3

The RSW lemma states conditions under which the crossing-probability of a rectangle of length $2n$ may be bounded below in terms of that of a rectangle of length n. Write an essay on the RSW lemma and one of its applications. Your essay should contain a clear statement and outline proof of the RSW lemma, together with an account of *either* its application in the exact calculation of a critical probability, *or* its use in the proof of Cardy's formula (accounts of *both* are not required and will gain no extra marks).

The emphasis should be more upon communication of the overall picture than giving the full details.

4

Define the random-cluster measure $\phi_{G,p,q}$ on a finite graph G, and the random-cluster measure $\phi^{\xi}_{\Lambda_n,p,q}$ on $\Lambda_n = \{-n, -n+1, \ldots, n\}^2 \subset \mathbb{Z}^2$ with boundary condition ξ.

Let $n \geqslant 1$, $p \in (0,1)$ and $q \geqslant 1$. Give a precise formulation of the monotonicity of $\phi^{\xi}_{\Lambda_n,p,q}$ in the boundary condition ξ, and prove it. [Any general result to which you refer should be stated clearly but need not be proved.]

Show that for a finite *planar* graph $G = (V, E)$ we have

$$\phi_{G,p_{sd}(q),q}(\omega) \propto \sqrt{q}^{k(\omega)+k(\overline{\omega}^*)}, \quad \omega \in \{0,1\}^E$$

where $p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$, $k(\omega)$ is the number of clusters in ω, and $k(\overline{\omega}^*)$ is the number of clusters in the dual configuration $\overline{\omega}^*$ of ω. [You may use Euler's formula without proof.]

END OF PAPER